Riemannian Geometry

Homework 9

Due on November 17

1. Consider the usual local coordinates (θ, φ) in $S^2 \subset \mathbb{R}^3$ defined by the parameterization $\phi : (0, \pi) \times (0, 2\pi) \to \mathbb{R}^3$ given by

$$\phi(\theta, \varphi) = (\sin\theta \cos\varphi, \sin\theta \sin\varphi, \cos\theta).$$

   (a) Using these coordinates, determine the expression of the Riemannian metric induced on S^2 by the Euclidean metric of $\mathbb{R}^3$.

   (b) Compute the Christoffel symbols for the Levi-Civita connection in these coordinates.

   (c) Show that the equator is the image of a geodesic.

   (d) Show that any rotation about an axis through the origin in $\mathbb{R}^3$ induces an isometry of S^2.

   (e) Show that the geodesics of S^2 traverse great circles.

   (f) Find a **geodesic triangle** (i.e. a triangle whose sides are images of geodesics) whose internal angles add up to $\frac{3\pi}{2}$.

   (g) Let $c : \mathbb{R} \to S^2$ be given by $c(t) = (\sin\theta_0 \cos t, \sin\theta_0 \sin t, \cos\theta_0)$, where $\theta_0 \in \left(0, \frac{\pi}{2}\right)$ (therefore c is not a geodesic). Let V be a vector field parallel along c such that $V(0) = \frac{\partial}{\partial\theta}$ ($\frac{\partial}{\partial\theta}$ is well defined at $(\sin\theta_0, 0, \cos\theta_0)$ by continuity). Compute the angle by which V is rotated when it returns to the initial point. (**Remark:** The angle you have computed is exactly the angle by which the oscillation plane of the **Foucault pendulum** rotates during a day in a place at latitude $\frac{\pi}{2} - \theta_0$, as it tries to remain fixed with respect to the stars in a rotating Earth).

   (h) Use this result to prove that no open set $U \subset S^2$ is isometric to an open set $W \subset \mathbb{R}^2$ with the Euclidean metric.

   (i) Given a geodesic $c : \mathbb{R} \to \mathbb{R}^2$ of $\mathbb{R}^2$ with the Euclidean metric and a point $p \notin c(\mathbb{R})$, there exists a unique geodesic $\widetilde{c} : \mathbb{R} \to \mathbb{R}^2$ (up to reparameterization) such that $p \in \widetilde{c}(\mathbb{R})$ and $c(\mathbb{R}) \cap \widetilde{c}(\mathbb{R}) = \varnothing$ (**parallel postulate**). Is this true in S^2?

2. **(Optional)** We introduce in $\mathbb{R}^3$, with the usual Euclidean metric $\langle \cdot, \cdot \rangle$, the connection ∇ defined in Cartesian coordinates (x^1, x^2, x^3) by

$$\Gamma^i_{jk} = \omega \varepsilon_{ijk},$$

where $\omega : \mathbb{R}^3 \to \mathbb{R}$ is a smooth function and

$$\varepsilon_{ijk} = \begin{cases} +1 & \text{if } (i,j,k) \text{ is an even permutation of } (1,2,3) \\ -1 & \text{if } (i,j,k) \text{ is an odd permutation of } (1,2,3) \\ 0 & \text{otherwise.} \end{cases}$$

Show that:

(a) ∇ is compatible with $\langle \cdot, \cdot \rangle$;

(b) the geodesics of ∇ are straight lines;

(c) the torsion of ∇ is not zero in all points where $\omega \neq 0$ (therefore ∇ is not the Levi-Civita connection unless $\omega \equiv 0$);

(d) the parallel transport equation is

$$\dot{V}^i + \sum_{j,k=1}^{3} \omega \varepsilon_{ijk} \dot{x}^j V^k = 0 \Leftrightarrow \dot{V} + \omega(\dot{x} \times V) = 0$$

(where $\times$ is the cross product in $\mathbb{R}^3$); therefore, a vector parallel along a straight line rotates about it with angular velocity $-\omega \dot{x}$.

3. **(Optional)** Let $(M, \langle \cdot, \cdot \rangle)$ be a Riemannian manifold with Levi-Civita connection $\widetilde{\nabla}$, and let $(N, \langle\langle \cdot, \cdot \rangle\rangle)$ be a submanifold with the induced metric and Levi-Civita connection ∇.

(a) Show that

$$\nabla_X Y = \left(\widetilde{\nabla}_{\widetilde{X}} \widetilde{Y} \right)^\top$$

for all $X, Y \in \mathfrak{X}(N)$, where $\widetilde{X}, \widetilde{Y}$ are any extensions of X, Y to $\mathfrak{X}(M)$ and $^\top : TM|_N \to TN$ is the orthogonal projection.

(b) Use this result to indicate curves that are, and curves that are not, geodesics of the following surfaces in $\mathbb{R}^3$:

i. the sphere S^2;

ii. the torus of revolution;

iii. the surface of a cone;

iv. a general surface of revolution.

(c) Show that if two surfaces in $\mathbb{R}^3$ are tangent along a curve, then the parallel transport of vectors along this curve in both surfaces coincides.

(d) Use this result to compute the angle $\Delta\theta$ by which a vector V is rotated when it is parallel transported along a circle on the sphere. (**Hint:** Consider the cone which is tangent to the sphere along the circle; notice that the cone minus a ray through the vertex is isometric to an open set of the Euclidean plane).