Riemannian Geometry

Homework 6

Due on October 27

1. Given the differential forms

$$\begin{aligned}
\alpha &= xdx + ydy \in \Omega^1(\mathbb{R}^2)\\
\beta &= -\frac{y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy \in \Omega^1(\mathbb{R}^2 \setminus \{0\})\\
\omega &= e^{xz}dx + x\cos zdy + y^2dz \in \Omega^1(\mathbb{R}^3)\\
\eta &= xdx \wedge dy - zdx \wedge dz + xyzdy \wedge dz \in \Omega^2(\mathbb{R}^3)\\
\zeta &= dx^1 \wedge dx^2 + \ldots + dx^{2n-1} \wedge dx^{2n} \in \Omega^2(\mathbb{R}^{2n})
\end{aligned}$$

and the smooth maps

$$\begin{array}{ccc}
f : \mathbb{R} \to \mathbb{R}^2 & g : (0, +\infty) \times (0, 2\pi) \to \mathbb{R}^2 & h : \mathbb{R}^3 \to \mathbb{R}^3\\
f(t) = (t, t^2) & g(r, \theta) = (r\cos\theta, r\operatorname{sen}\theta) & h(u, v, w) = (uv, vw, uw)
\end{array}$$

compute:

(a) $\alpha \wedge \beta$, $\omega \wedge \eta$, $\eta \wedge \eta$;

(b) $\zeta \wedge \ldots \wedge \zeta$ (wedge product with n factors);

(c) $d\alpha, d\beta, d\omega, d\eta, d\zeta$.

(d) $f^*\alpha, g^*\alpha, g^*\beta, h^*\eta$.

2. Given the vector isomorphisms $i_1 : \mathfrak{X}(\mathbb{R}^3) \to \Omega^1(\mathbb{R}^3)$ and $i_2 : \mathfrak{X}(\mathbb{R}^3) \to \Omega^2(\mathbb{R}^3)$ defined by

$$\begin{aligned}
i_1\left(X^1\frac{\partial}{\partial x} + X^2\frac{\partial}{\partial y} + X^3\frac{\partial}{\partial z}\right) &= X^1dx + X^2dy + X^3dz\\
i_2\left(X^1\frac{\partial}{\partial x} + X^2\frac{\partial}{\partial y} + X^3\frac{\partial}{\partial z}\right) &= X^1dy \wedge dz + X^2dz \wedge dx + X^3dx \wedge dy
\end{aligned}$$

show that for $f \in C^\infty(\mathbb{R}^3)$ and $X, Y \in \mathfrak{X}(\mathbb{R}^3)$:

(a) $df = i_1(\nabla f)$, where $\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$ is the **gradient** of f;

(b) $d(i_1(X)) = i_2(\nabla \times X)$, where $\nabla \times X = \left(\frac{\partial X^3}{\partial y} - \frac{\partial X^2}{\partial z}, \frac{\partial X^1}{\partial z} - \frac{\partial X^3}{\partial x}, \frac{\partial X^2}{\partial x} - \frac{\partial X^1}{\partial y}\right)$ is the **curl** of X;

(c) $d(i_2(X)) = (\nabla \cdot X)\, dx \wedge dy \wedge dz$, where $\nabla \cdot X = \dfrac{\partial X^1}{\partial x} + \dfrac{\partial X^2}{\partial x} + \dfrac{\partial X^3}{\partial z}$ is the **divergence** of X;

(d) $d(df) = 0$ implies $\nabla \times (\nabla f) = 0$;

(e) $d(d(i_1(X))) = 0$ implies $\nabla \cdot (\nabla \times X) = 0$;

(f) $\nabla \times (fX) = (\nabla f) \times X + f(\nabla \times X)$;

(g) $\nabla \cdot (X \times Y) = (\nabla \times X) \cdot Y - X \cdot (\nabla \times Y)$.

3. **(Optional)** A k-form ω is called **closed** if $d\omega = 0$. If it exists a $(k-1)$-form β such that $\omega = d\beta$ then ω is called **exact**. Note that every exact form is closed. Let Z^k be the set of all closed k-forms on M and define a relation between forms on Z^k as follows: $\alpha \sim \beta$ if and only if they differ by an exact form, that is, if $\beta - \alpha = d\theta$ for some $(k-1)$-form θ.

   (a) Show that this relation is an equivalence relation.

   (b) Let $H^k(M)$ be the corresponding set of equivalence classes (called the k-dimensional **de Rham cohomology space** of M). Show that addition and scalar multiplication of forms define indeed a vector space structure on $H^k(M)$.

   (c) Let $f : M \to N$ be a smooth map. Show that:

      i. the pull-back f^* carries closed forms to closed forms and exact forms to exact forms;

      ii. if $\alpha \sim \beta$ on N then $f^*\alpha \sim f^*\beta$ on M;

      iii. f^* induces a linear map on cohomology $f^\sharp : H^k(N) \to H^k(M)$ naturally defined by $f^\sharp[\omega] = [f^*\omega]$;

      iv. if $g : L \to M$ is another smooth map, then $(f \circ g)^\sharp = g^\sharp \circ f^\sharp$.

   (d) Show that the dimension of $H^0(M)$ is equal to the number of connected components of M.

   (e) Show that $H^k(M) = 0$ for every $k > \dim M$.

4. **(Optional)** Let M be a manifold of dimension n, let U be an open subset of $\mathbb{R}^n$ and let ω be a k-form on $\mathbb{R} \times U$. Writing ω as

$$\omega = dt \wedge \sum_I a_I dx^I + \sum_J b_J dx^J,$$

where $I = (i_1, \ldots, i_{k-1})$ and $J = (j_1, \ldots, j_k)$ are increasing index sequences, $(x^1, \ldots, x^n)$ are coordinates in U and t is the coordinate in $\mathbb{R}$, consider the operator $\mathcal{Q}$ defined by

$$\mathcal{Q}(\omega)_{(t,x)} = \sum_I \left(\int_{t_0}^t a_I ds \right) dx^I,$$

which transforms k-forms ω in $\mathbb{R} \times U$ into $(k-1)$-forms.

(a) Let $f : V \to U$ be a diffeomorphism between open subsets of $\mathbb{R}^n$. Show that the induced diffeomorphism $\widetilde{f} := \mathrm{id} \times f : \mathbb{R} \times V \to \mathbb{R} \times U$ satisfies

$$\widetilde{f}^* \circ \mathcal{Q} = \mathcal{Q} \circ \widetilde{f}^*.$$

(b) Using (a), construct an operator $\mathcal{Q}$ which carries k-forms on $\mathbb{R} \times M$ into $(k-1)$-forms and, for any diffeomorphism $f : M \to N$, the induced diffeomorphism $\widetilde{f} := \mathrm{id} \times f : \mathbb{R} \times M \to \mathbb{R} \times N$ satisfies $\widetilde{f}^* \circ \mathcal{Q} = \mathcal{Q} \circ \widetilde{f}^*$. Show that this operator is linear.

(c) Considering the operator $\mathcal{Q}$ defined in (b) and the inclusion $i_{t_0} : M \to \mathbb{R} \times M$ of M at the "level" t_0, defined by $i_{t_0}(p) = (t_0, p)$, show that $\omega - \pi^* i_{t_0}^* \omega = d\mathcal{Q}\omega + \mathcal{Q}d\omega$, where $\pi : \mathbb{R} \times M \to M$ is the projection on M.

(d) Show that the maps $\pi^\sharp : H^k(M) \to H^k(\mathbb{R} \times M)$ and $i_{t_0}^\sharp : H^k(\mathbb{R} \times M) \to H(M)$ are inverses of each other (and so $H^k(M)$ is isomorphic to $H^k(\mathbb{R} \times M)$).

(e) Use (d) to show that, for $k > 0$ and $n > 0$, every closed k-form in $\mathbb{R}^n$ is exact, that is, $H^k(\mathbb{R}^n) = 0$ if $k > 0$.

(f) Use (d) to show that, if $f, g : M \to N$ are two **smoothly homotopic maps** between smooth manifolds (meaning that there exists a smooth map $H : \mathbb{R} \times M \to N$ such that $H(t_0, p) = f(p)$ and $H(t_1, p) = g(p)$ for some fixed $t_0, t_1 \in \mathbb{R}$), then $f^\sharp = g^\sharp$.

(g) We say that M is **contractible** if the identity map id : $M \to M$ is smoothly homotopic to a constant map. Show that $\mathbb{R}^n$ is contractible.

(h) Let M be a contractible smooth manifold. Show that every closed form on M is exact, that is, $H^k(M) = 0$ for all $k > 0$ (**Poincaré Lemma**).