Riemannian Geometry

Homework 4

Due on October 13

1. (a) Let N be a differentiable manifold, $M \subset N$ a submanifold and $X, Y \in \mathfrak{X}(N)$ vector fields tangent to M, i.e., such that $X_p, Y_p \in T_pM$ for all $p \in M$. Show that $[X, Y]$ is also tangent to M, and that its restriction to M coincides with the Lie bracket of the restrictions of X and Y to M.

   (b) Use (a) to argue that the Lie bracket on the Lie algebra of any Lie subgroup of $GL(n)$ is the usual matrix commutator.

2. We can identify each point in

$$H = \{(x, y) \in \mathbb{R}^2 \mid y > 0\}$$

with the invertible affine map $h : \mathbb{R} \to \mathbb{R}$ given by $h(t) = yt + x$. The set of all such maps is a group under composition; consequently, our identification induces a group structure on H.

   (a) Show that the induced group operation is given by

$$(x, y) \cdot (z, w) = (yz + x, yw),$$

   and that H, with this group operation, is a Lie group.

   (b) Show that the derivative of the left translation map $L_{(x,y)} : H \to H$ at a point $(z, w) \in H$ is represented in the above coordinates by the matrix

$$\left(dL_{(x,y)}\right)_{(z,w)} = \begin{pmatrix} y & 0 \\ 0 & y \end{pmatrix}.$$

   Conclude that the left-invariant vector field $X^V \in \mathfrak{X}(H)$ determined by the vector

$$V = \xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y} \in \mathfrak{h} \equiv T_{(0,1)}H \qquad (\xi, \eta \in \mathbb{R})$$

   is given by

$$X^V_{(x,y)} = \xi y \frac{\partial}{\partial x} + \eta y \frac{\partial}{\partial y}.$$

   (c) Given $V, W \in \mathfrak{h}$, compute $[V, W]$.

   (d) Determine the flow of the vector field X^V, and give an expression for the exponential map $\exp : \mathfrak{h} \to H$.

(e) Confirm your results by first showing that H is the subgroup of $GL(2)$ formed by the matrices

$$\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}$$

with $y > 0$.

3. **(Optional)** Consider the vector field $X \in \mathfrak{X}(\mathbb{R}^2)$ defined by

$$X = \sqrt{x^2 + y^2}\frac{\partial}{\partial x}.$$

(a) Show that the flow of X defines a free action of $\mathbb{R}$ on $M = \mathbb{R}^2 \setminus \{0\}$.

(b) Describe the topological quotient space $M/\mathbb{R}$. Is the action above proper?

4. **(Optional)**

(a) Show that $\mathbb{R} \cdot SU(2)$ is a four dimensional real linear subspace of $\mathcal{M}_{2\times 2}(\mathbb{C})$, closed under matrix multiplication, with basis

$$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad i = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix},$$

$$j = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad k = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},$$

satisfying $i^2 = j^2 = k^2 = ijk = -1$. Therefore this space can be identified with the **quaternions**. Show that $SU(2)$ can be identified with the quaternions of Euclidean norm equal to 1, and is therefore diffeomorphic to S^3.

(b) Show that if $n \in \mathbb{R}^3$ is a unit vector, which we identify with a quaternion with zero real part, then

$$\exp\left(\frac{n\theta}{2}\right) = 1\cos\left(\frac{\theta}{2}\right) + n\sin\left(\frac{\theta}{2}\right)$$

is also a unit quaternion.

(c) Again identifying $\mathbb{R}^3$ with quaternions with zero real part, show that the map

$$\begin{aligned} \mathbb{R}^3 &\to \mathbb{R}^3 \\ v &\mapsto \exp\left(\frac{n\theta}{2}\right) \cdot v \cdot \exp\left(-\frac{n\theta}{2}\right) \end{aligned}$$

is a rotation by an angle θ about the axis defined by n.

(d) Show that there exists a surjective homomorphism $F : SU(2) \to SO(3)$, and use this to conclude that $SU(2)$ is the universal covering of $SO(3)$.

(e) What is the fundamental group of $SO(3)$?