Riemannian Geometry

Homework 3

Due on October 6

1. Let $X, Y \in \mathfrak{X}(\mathbb{R}^3)$ be the vector fields defined by
$$X = y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y} \qquad \text{and} \qquad Y = z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z},$$
where (x, y, z) are the usual Cartesian coordinates.
   (a) Compute the Lie bracket $[X, Y]$.
   (b) Compute the flows ψ_t and ϕ_t of X and Y.
   (c) Check explicitly that $\psi_{\frac{\pi}{2}} \circ \phi_{\frac{\pi}{2}} \neq \phi_{\frac{\pi}{2}} \circ \psi_{\frac{\pi}{2}}$.

2. Let $f : M \to N$ be a diffeomorphism between smooth manifolds. Show that $f_*[X, Y] = [f_*X, f_*Y]$ for every $X, Y \in \mathfrak{X}(M)$.

3. **(Optional)** Let $X, Y \in \mathfrak{X}(M)$ be two complete vector fields with flows ψ, ϕ. Show that:
   (a) given a diffeomorphism $f : M \to M$, we have $f_*X = X$ if and only if $f \circ \psi_t = \psi_t \circ f$ for all $t \in \mathbb{R}$;
   (b) $\psi_t \circ \phi_s = \phi_s \circ \psi_t$ for all $s, t \in \mathbb{R}$ if and only if $[X, Y] = 0$.

4. **(Optional)** For two vector fields $X, Y \in \mathfrak{X}(M)$ we define the **Lie derivative** of Y in the direction of X as
$$L_X Y := \frac{d}{dt}((\psi_{-t})_* Y)_{|t=0},$$
where $\{\psi_t\}_{t \in I}$ is the local flow of X. Show that:
   (a) $L_X Y = [X, Y]$;
   (b) $L_X[Y, Z] = [L_X Y, Z] + [Y, L_X Z]$, for $X, Y, Z \in \mathfrak{X}(M)$;
   (c) $L_X \circ L_Y - L_Y \circ L_X = L_{[X,Y]}$.

5. **(Optional)** Let $f : M \to N$ be a differentiable map between smooth manifolds and consider submanifolds $V \subset M$ and $W \subset N$. Show that if $f(V) \subset W$ then $f : V \to W$ is also a differentiable map.