Riemannian Geometry

Homework 2

Due on September 29

1. Let M be a smooth manifold, $p \in M$ and $v \in T_pM$. Show that if v can be written as $v = \sum_{i=1}^n a^i (\frac{\partial}{\partial x^i})_p$ and $v = \sum_{i=1}^n b^i (\frac{\partial}{\partial y^i})_p$ for two basis associated to two different local charts around p, then
$$b^j = \sum_{i=1}^n \left(\frac{\partial y^j}{\partial x^i}(x(p)) \right) a^i.$$

2. Let $f : M \to N$ and $g : N \to P$ be two differentiable maps. Show that:
   (a) $g \circ f : M \to P$ is also differentiable;
   (b) $(d(g \circ f))_p = (dg)_{f(p)} \circ (df)_p$ for any $p \in M$.

3. Consider the n-sphere
$$S^n = \left\{ x \in \mathbb{R}^{n+1} : (x^1)^2 + \cdots (x^{n+1})^2 = 1 \right\}.$$
   Show that:
   (a) S^n is an n-dimensional submanifold of $\mathbb{R}^{n+1}$;
   (b) $T_xS^n = \left\{ v \in \mathbb{R}^{n+1} : \langle x, v \rangle = 0 \right\}$, where $\langle \cdot, \cdot \rangle$ is the usual inner product on $\mathbb{R}^n$ and we are using the natural identification $T_x\mathbb{R}^n \cong \mathbb{R}^n$.

4. **(Optional)** Let $f : M \to N$ be an injective immersion. Show that if M is compact then f is an embedding.

5. **(Optional)** Show that the map $f : \mathbb{R}P^2 \to \mathbb{R}^4$ given by
$$f([x, y, z]) = \frac{1}{x^2 + y^2 + z^2}(xz, yz, xy, x^2 - y^2)$$
   is an embedding.

6. **(Optional)** Show that the **orthogonal group**
$$O(n) = \{ A \in \mathcal{M}_{n \times n} \mid A^t A = I \}$$
   is an $\frac{n(n-1)}{2}$-dimensional submanifold of $\mathcal{M}_{n \times n} \cong \mathbb{R}^{n^2}$.

7. **(Optional)** Show that the boundary ∂Q of the cube $Q = [-1, 1] \times [-1, 1] \times [-1, 1]$ is **not** a submanifold of $\mathbb{R}^3$.