Riemannian Geometry

Homework 13

Due on December 15

1. Use the second fundamental form to compute the Gauss curvature of the following surfaces in $\mathbb{R}^3$:

   (a) The paraboloid $z = \frac{1}{2}\left(x^2 + y^2\right)$.

   (b) The saddle surface $z = xy$.

2. Let $(M, \langle \cdot, \cdot \rangle)$ be a Riemannian manifold. A submanifold $N \subset M$ is said to be **totally geodesic** if the the geodesics of N are geodesics of M. Show that:

   (a) N is totally geodesic if and only if $B \equiv 0$, where B is the second fundamental form of N.

   (b) If N is the set of fixed points of an isometry then N is totally geodesic. Use this result to give examples of totally geodesic submanifolds of $\mathbb{R}^n$, S^n and H^n.

3. **(Optional)** Let $(M, \langle \cdot, \cdot \rangle)$ be a Riemannian manifold, p a point in M and Π a section of T_pM. For $B_\varepsilon(p) := \exp_p(B_\varepsilon(0))$ a normal ball around p consider the set $N_p := \exp_p(B_\varepsilon(0) \cap \Pi)$. Show that:

   (a) The set N_p is a 2-dimensional submanifold of M formed by the segments of geodesics in $B_\varepsilon(p)$ which are tangent to Π at p.

   (b) If in N_p we use the metric induced by the metric in M, the sectional curvature $K^M(\Pi)$ is equal to the Gauss curvature of the 2-manifold N_p.

4. **(Optional)** If $N \subset \mathbb{R}^{n+1}$ is an oriented hypersurface we define its **Gauss map** $g : N \to S^n$ to be the map such that $g(p)$ is the unit normal vector compatible with the orientation. Since $g(p)$ is normal to T_pN, we can identify the tangent spaces T_pN and $T_{g(p)}S^n$ to obtain a well-defined map $(dg)_p : T_pN \to T_pN$. Show that:

   (a) $(dg)_p = -S_{g(p)}$, where $S_{g(p)} : T_pN \to T_pN$ is the symmetric linear operator such that $\langle\langle S_{g(p)}X_p, Y_p \rangle\rangle = \langle B(X_p, Y_p), g(p) \rangle$ (here B is the second fundamental form, $\langle \cdot, \cdot \rangle$ is the Euclidean inner product and $\langle\langle \cdot, \cdot \rangle\rangle$ is the induced metric).

   (b) If the Gauss curvature $K(p)$ of N at p does not vanish then

   $$|K(p)| = \lim_{D \to p} \frac{\text{vol}(g(D))}{\text{vol}(D)},$$

   where D is a neighborhood of p in N whose diameter tends to zero.