Riemannian Geometry

Homework 12

Due on December 12

1. Let M be an oriented Riemannian 2-manifold and let p be a point in M. Let D be a neighborhood of p in M homeomorphic to a disc, with a smooth boundary ∂D. Consider a point $q \in \partial D$ and a unit vector $X_q \in T_qM$. Let X be the parallel transport of X_q along ∂D in the positive direction. When X returns to q it makes an angle $\Delta\theta$ with the initial vector X_q. Using fields of positively oriented orthonormal frames $\{E_1, E_2\}$ and $\{F_1, F_2\}$ such that $F_1 = X$, show that
$$\Delta\theta = \int_D K.$$
Conclude that the Gauss curvature of M at p satisfies
$$K(p) = \lim_{D \to p} \frac{\Delta\theta}{\mathrm{vol}(D)}.$$

2. **Gauss-Bonnet Theorem for manifolds with boundary:** Let M be a compact, oriented, 2-dimensional manifold with boundary and let X be a vector field in M **transverse** to ∂M (i.e., such that $X_p \notin T_p\partial M$ for all $p \in \partial M$), with isolated singularities $p_1, \ldots, p_k \in M \backslash \partial M$. Prove that
$$\int_M K + \int_{\partial M} k_g = 2\pi \sum_{i=1}^{k} I_{p_i}$$
for any Riemannian metric on M, where K is the Gauss curvature of M and k_g is the geodesic curvature of ∂M.

3. Prove **Schur's Theorem:** Any be a connected isotropic Riemannian manifold M of dimension $n \geq 3$ has constant curvature. (**Hint:** Use the structure equations to show that $dK = 0$).

4. **(Optional)** Let (M, g) be a 2-dimensional Riemannian manifold and $\Delta \subset M$ a **geodesic triangle**, i.e., an open set homeomorphic to an Euclidean triangle whose sides are images of geodesic arcs. Let α, β, γ be the inner angles of Δ, i.e., the angles between the geodesics at the intersection points contained in $\partial\Delta$. Prove that for small enough Δ one has
$$\alpha + \beta + \gamma = \pi + \int_\Delta K,$$
where K is the Gauss curvature of M, using:
   (a) the fact that $\int_\Delta K$ is the angle by which a vector parallel-transported once around $\partial\Delta$ rotates;
   (b) the Gauss-Bonnet Theorem for manifolds with boundary.

(**Remark:** We can use this result to give another geometric interpretation of the Gauss curvature: $K(p) = \lim_{\Delta \to p} \frac{\alpha+\beta+\gamma-\pi}{\mathrm{vol}(\Delta)}$).

5. **(Optional)** Let (M, g) be a compact orientable 2-dimensional Riemannian manifold, with positive Gauss curvature. Show that any two non-self-intersecting closed geodesics must intersect each other.

6. **(Optional)** Let (M, g) be a simply connected 2-dimensional Riemannian manifold with nonpositive Gauss curvature. Show that any two geodesics intersect at most in one point. (**Hint:** Note that if two geodesics intersected in more than one point then there would exist a **geodesic biangle**, i.e., an open set homeomorphic to a disc whose boundary is formed by the images of two geodesic arcs).

7. **(Optional)** Show that the open half-space

$$H = \{(x^1, \ldots x^n) \in \mathbb{R}^n \mid x^n > 0\}$$

equipped with the Riemannian metric

$$g = \frac{1}{(x^n)^2} \left(dx^1 \otimes dx^1 + \ldots + dx^n \otimes dx^n \right)$$

has constant sectional curvature $K = -1$.

8. **(Optional)** Let H^2 be the hyperbolic plane. Show that:

   (a) The formula
   $$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z \ := \ \frac{az+b}{cz+d} \qquad\qquad (ad - bc = 1)$$
   defines an action of $PSL(2, \mathbb{R}) := SL(2, \mathbb{R})/\{\pm \mathrm{id}\}$ on H^2 by orientation-preserving isometries;

   (b) for any two geodesics $c_1, c_2 : \mathbb{R} \to H^2$, parameterized by the arclength, there exists $g \in PSL(2, \mathbb{R})$ such that $c_1(s) = g \cdot c_2(s)$ for all $s \in \mathbb{R}$;

   (c) if $f : H^2 \to H^2$ is an orientation-preserving isometry then it must be a holomorphic function. Conclude that all orientation-preserving isometries are of the form $f(z) = g \cdot z$ for some $g \in PSL(2, \mathbb{R})$.