Riemannian Geometry

Homework 11

Due on December 2

1. Compute the Gauss curvature of:

   (a) The sphere S^2 with the standard metric, given in spherical coordinates (θ, φ) by
   $$g = d\theta \otimes d\theta + \sin^2\theta \, d\varphi \otimes d\varphi.$$

   (b) The hyperbolic plane, that is, the upper half-plane
   $$H = \{(x, y) \in \mathbb{R}^2 \mid y > 0\}$$
   with the metric
   $$g = \frac{1}{y^2}(dx \otimes dx + dy \otimes dy).$$

2. Show that $\|X_p\|^2\|Y_p\|^2 - \langle X_p, Y_p\rangle^2$ gives us the square of the area of the parallelogram in T_pM spanned by X_p, Y_p. Conclude that the sectional curvature does not depend on the choice of the linearly independent vectors X_p, Y_p, that is, when we change of basis on Π, both $R(X_p, Y_p, X_p, Y_p)$ and $\|X_p\|^2\|Y_p\|^2 - \langle X_p, Y_p\rangle^2$ change by the square of the determinant of the change of basis matrix.

3. **(Optional)** Recall that if G is a Lie group endowed with a bi-invariant Riemannian metric, ∇ is the Levi-Civita connection and X, Y are two left-invariant vector fields then
   $$\nabla_X Y = \frac{1}{2}[X, Y].$$
   Show that if Z is also left-invariant, then
   $$R(X, Y)Z = \frac{1}{4}[Z, [X, Y]].$$

4. **(Optional)** Consider the metric
   $$g = dr \otimes dr + f^2(r) d\theta \otimes d\theta$$
   on $M = I \times S^1$, where r is a local coordinate on $I \subset \mathbb{R}$ and θ is the usual angular coordinate on S^1.

   (a) Compute the Gauss curvature of this metric.

   (b) For which functions $f(r)$ is the Gauss curvature constant?

5. **(Optional)** Consider the metric

$$g = A^2(r)dr \otimes dr + r^2 d\theta \otimes d\theta + r^2 \sin^2\theta \, d\varphi \otimes d\varphi$$

on $M = I \times S^2$, where r is a local coordinate on $I \subset \mathbb{R}$ and (θ, φ) are spherical local coordinates on S^2.

(a) Compute the Ricci tensor and the scalar curvature of this metric.

(b) What happens when $A(r) = (1 - r^2)^{-\frac{1}{2}}$ (that is, when M is locally isometric to S^3)?

(c) And when $A(r) = (1 + r^2)^{-\frac{1}{2}}$ (that is, when M is locally isometric to the **hyperbolic** 3-**space**)?

(d) For which functions $A(r)$ is the scalar curvature constant?