Riemannian Geometry

Homework 1

Due on September 22

1. A **triangulation** of a 2-dimensional topological manifold M is a decomposition of M into a finite number of triangles (i.e. subsets homeomorphic to triangles in $\mathbb{R}^2$) such that the intersection of any two triangles is either a common edge, a common vertex or empty (it is possible to prove that such a triangulation always exists). The **Euler characteristic** of M is
$$\chi(M) := V - E + F,$$
where V, E and F are the number of vertices, edges and faces of a given triangulation (it can be shown that this is well defined, i.e. does not depend on the choice of triangulation). Show that:

   (a) adding a vertex to a triangulation does not change $\chi(M)$;

   (b) $\chi(S^2) = 2$;

   (c) $\chi(T^2) = 0$;

   (d) $\chi(K^2) = 0$;

   (e) $\chi(\mathbb{R}P^2) = 1$;

   (f) $\chi(M \# N) = \chi(M) + \chi(N) - 2$;

   (g) there exists an infinite number of non-homeomorphic topological manifolds of dimension 2.

2. The **real projective plane** $\mathbb{R}P^2$ is the set of lines through the origin in $\mathbb{R}^3$. This space can be defined as the quotient space of $\mathbb{R}^3 \setminus \{(0,0,0)\}$ by the equivalence relation $\sim$ which identifies points on the same line through the origin, and is a topological manifold with parameterizations $\varphi_1, \varphi_2, \varphi_3 : \mathbb{R}^2 \to \mathbb{R}P^2$ given by
$$\varphi_1(p,q) = [1,p,q]; \qquad \varphi_2(s,t) = [s,1,t]; \qquad \varphi_3(u,v) = [1,u,v].$$
Show that:

   (a) $\{\varphi_1, \varphi_2, \varphi_3\}$ is a differentiable atlas for $\mathbb{R}P^2$;

   (b) the map $f : S^2 \to \mathbb{R}P^2$ given by $f(x,y,z) = [x,y,z]$ is differentiable (where we use the differentiable structure defined on S^2 by the stereographic projections).

3. **(Optional)**

   (a) Show that one can compute the Euler characteristic by using decompositions into squares, and check it explicitly for the examples in Exercise 1.

   (b) By using decompositions into cubes, define and compute the Euler characteristic of S^3 and T^3.

4. **(Optional)** Let M be the disjoint union of $\mathbb{R}$ with a point p and consider the maps $\varphi_i : \mathbb{R} \to M$ $(i = 1, 2)$ defined by $\varphi_i(x) = x$ if $x \in \mathbb{R}\backslash\{0\}$, $\varphi_1(0) = 0$ and $\varphi_2(0) = p$. Show that:

   (a) the maps $\varphi_i^{-1} \circ \varphi_j$ are differentiable on their domains;

   (b) if we consider an atlas formed by $\{(\mathbb{R}, \varphi_1), (\mathbb{R}, \varphi_2)\}$, the corresponding topology will not satisfy the Hausdorff axiom.

5. **(Optional)** Show that a differentiable map $f : M \to N$ between two differentiable manifolds is continuous.