Differential Geometry of Curves and Surfaces

Homework 12

Due on December 12

1. Let A be a **geodesic triangle** (that is, a simply connected domain with 3 corners whose boundary is the union of the images of 3 geodesics) on an oriented Riemannian surface, and let α, β and γ be its internal angles. Show that
$$\alpha + \beta + \gamma = \pi + \int_A K\theta^1 \wedge \theta^2,$$
where $\{\theta^1, \theta^2\}$ is dual to a positive orthonormal frame.

2. Let $\{\mathbf{e}_1, \mathbf{e}_2\}$ be a positive orthonormal frame on an oriented Riemannian surface, and let A be a simply connected domain. Suppose that $\mathbf{V} : [t_0, t_1] \to \mathbb{R}^2$ is a unit vector field which is parallel along the closed regular curve $\mathbf{c} : [t_0, t_1] \to \partial A$ which parameterizes ∂A in the direction induced by the orientation of A. Show that:

   (a) If we write
   $$\mathbf{V}(t) = \cos(\varphi(t))\mathbf{e}_1 + \sin(\varphi(t))\mathbf{e}_2$$
   then
   $$\frac{d\varphi}{dt}(t) = \omega^1_2(\dot{\mathbf{c}}(t)),$$
   where ω^1_2 is the connection form associated to $\{\mathbf{e}_1, \mathbf{e}_2\}$.

   (b) The angle α between $\mathbf{V}(t_1)$ and $\mathbf{V}(t_0)$ at the point $\mathbf{c}(t_0) = \mathbf{c}(t_1)$ is
   $$\alpha = \int_A K\theta^1 \wedge \theta^2,$$
   where $\{\theta^1, \theta^2\}$ is the co-frame dual to $\{\mathbf{e}_1, \mathbf{e}_2\}$.

   (c) If A is the simply connected domain
   $$A = \{(x, y, z) \in S^2 : z \geq z_0\}$$
   on the unit 2-sphere $S^2 \subset \mathbb{R}^3$ then $\alpha = 2\pi(1 - z_0)$.
   Remark: This is the angle by which the oscillation plane of a Foucault pendulum rotates after 24 hours on a location with latitude $\lambda = \arcsin z_0$.