Differential Geometry of Curves and Surfaces

Homework 10

Due on November 28

1. Consider the Riemannian metric on the plane written in polar coordinates (r, θ) as

$$ds^2 = dr^2 + f^2(r)d\theta^2.$$

Assume that ds^2 can be extended to the whole plane, and that f has the Taylor expansion around $r = 0$ given by

$$f(r) = r - \frac{k}{6}r^3 + O(r^5).$$

Show that:

(a) This Riemannian metric can be written as

$$ds^2 = \left(\theta^1\right)^2 + \left(\theta^2\right)^2, \qquad \theta^1 = dr, \qquad \theta^2 = f(r)d\theta.$$

(b) The connection form associated to $\{\theta^1, \theta^2\}$ is

$$\omega_2{}^1 = -f'(r)d\theta.$$

(c) The Gauss curvature of this Riemannian surface is

$$K(r, \theta) = -\frac{f''(r)}{f(r)}.$$

(d) The Gauss curvature at the origin is

$$\lim_{r \to 0} K(r, \theta) = k.$$

(e) The length of the circle of radius r centered at the origin has the Taylor expansion

$$l = 2\pi r \left(1 - \frac{k}{6}r^2 + O(r^4)\right).$$

(f) The area of the circle of radius r centered at the origin has the Taylor expansion

$$A = \pi r^2 \left(1 - \frac{k}{12}r^2 + O(r^4)\right).$$