Differential Geometry of Curves and Surfaces

2025/2026

2^{nd} Exam - 6 February 2026 - 10:30
Duration: 2 hours

1. Let $\mathbf{c} : (a, b) \to \mathbb{R}^3$ be a regular space curve with nonvanishing curvature parameterized by arclength, and consider the parameterization $\mathbf{g} : (a, b) \times \mathbb{R}^+ \to S \subset \mathbb{R}^3$ given by

$$\mathbf{g}(s, t) = \mathbf{c}(s) + t\,\mathbf{c}'(s)\,.$$

(2/20) (a) Check that the first fundamental form associated to $\mathbf{g}$ is

$$\mathbf{I} = \left(1 + t^2\kappa^2(s)\right) ds^2 + 2ds dt + dt^2\,,$$

where $\kappa(s)$ is the curvature of $\mathbf{c}$.

(2/20) (b) Show that the second fundamental form associated to $\mathbf{g}$ is

$$\mathbf{II} = -t\kappa(s)\tau(s)ds^2\,,$$

where $\tau(s)$ is the torsion of $\mathbf{c}$.

(2/20) (c) Conclude that the Gauss curvature and the mean curvature of S are

$$K(s, t) = 0 \qquad \text{and} \qquad H(t, s) = -\frac{\tau(s)}{2t\kappa(s)}\,.$$

What are the principal curvatures?

2. Consider the Riemannian metric defined in the open set $\mathbb{R}^2 \setminus \{(0, 0)\}$ by the expression

$$ds^2 = \frac{1}{(1 + r^2)^2}\left(4dr^2 + r^2 d\theta^2\right)\,,$$

given in polar coordinates $(r, \theta) \in \mathbb{R}^+ \times (0, 2\pi)$ (as usual the polar coordinates miss the positive x-axis).

(2/20) (a) Check that the Gaussian curvature is $K = 1$.

(2/20) (b) Prove that the curve $r = 1$ is the image of a geodesic.

(2/20) (c) Is this Riemannian surface isometric to some region of the unit sphere S^2? Why or why not?

(2/20) **3.** Compute, using your choice of orientation, the integral

$$\int_{S^2} x dy \wedge dz + y dx \wedge dz + z dx \wedge dy \, .$$

(2/20) **4.** Prove that there is no compact surface $S \subset \mathbb{R}^3$ whose Gauss curvature is strictly negative.

(2/20) **5.** Recall that the hyperbolic plane is the Riemannian surface corresponding to the first fundamental form

$$ds^2 = \frac{1}{y^2}\left(dx^2 + dy^2\right)$$

on the upper half-plane $H = \mathbb{R} \times \mathbb{R}^+$. Prove that if $\mathbf{c} : [a, b] \to H$ is a simple closed curve parameterized by arclength with geodesic curvature $\kappa_g(s)$ then

$$\int_a^b |\kappa_g(s)| \, ds \geq 2\pi \, .$$

Is this true for simple closed curves on the 2-sphere S^2?

(2/20) **6.** Let $f, g : U \subset \mathbb{C} \to \mathbb{C}$ be Weierstrass-Enneper data for a minimal surface S such that g is a bijection. Prove that

$$\int_S K = -4\pi \, .$$

Does this mean that S is homeomorphic to the double torus?