Differential Geometry of Curves and Surfaces

2024/2025

1st Exam - 20 January 2025 - 10:30
Duration: 2 hours

(2/20) **1.** Let $\mathbf{c} : [a,b] \to \mathbb{R}^3$ be a regular space curve satisfying $\|\mathbf{c}(t)\| = R$ for all $t \in [a,b]$, where $R > 0$ is a constant. Prove that the curvature of $\mathbf{c}$ at each point is at least $1/R$. Moreover, prove that if the curvature of $\mathbf{c}$ is constant equal to $1/R$ then $\mathbf{c}$ is a planar curve.

2. The **pseudosphere** is the surface of revolution parameterized by $\mathbf{g} : \mathbb{R}^+ \times (0, 2\pi) \to \mathbb{R}^3$ defined by

$$\mathbf{g}(u,\varphi) = \left(\frac{\cos\varphi}{\cosh u}, \frac{\sin\varphi}{\cosh u}, u - \tanh u\right)$$

(where as usual the parametrization misses the curve corresponding to $\varphi = 0$).

(2/20) (a) Show that the first fundamental form corresponding to this parameterization is

$$\mathbf{I} = \tanh^2 u \, du^2 + \frac{1}{\cosh^2 u} d\varphi^2.$$

(2/20) (b) Compute the total area of the pseudosphere.

(2/20) (c) Prove that the Gauss curvature is $K = -1$ (hence the name "pseudosphere").

(2/20) (d) Is the pseudosphere a minimal surface? Why or why not?

(2/20) (e) Show that the set $\{\mathbf{g}(u,\pi) : u \in \mathbb{R}^+\}$ is the image of a geodesic.

(2/20) (f) Does the pseudosphere contain a geodesic triangle with area larger than π?

3. Let $S \subset \mathbb{R}^n$ be a two-dimensional manifold, let $\mathbf{g} : U \subset \mathbb{R}^2 \to S$ be a parameterization, and let $\mathbf{e}_1, \dots, \mathbf{e}_n : U \to \mathbb{R}^n$ be a local orthonormal frame such that $\{\mathbf{e}_1(u,v), \mathbf{e}_2(u,v)\}$ is a basis for $T_{\mathbf{g}(u,v)}S$. Define the one-forms θ^1 and θ^2 as

$$d\mathbf{g} = \theta^1 \mathbf{e}_1 + \theta^2 \mathbf{e}_2,$$

the one-forms $\omega_i{}^j$ by the equations

$$d\mathbf{e}_i = \sum_{j=1}^{n} \omega_i{}^j \mathbf{e}_j, \qquad i = 1, \dots, n,$$

and the functions $b_{11}^{(j)}$, $b_{12}^{(j)}$, $b_{21}^{(j)}$ and $b_{22}^{(j)}$ as

$$\begin{cases} {\omega_1}^j = b_{11}^{(j)}\theta^1 + b_{12}^{(j)}\theta^2 \\ {\omega_2}^j = b_{21}^{(j)}\theta^1 + b_{22}^{(j)}\theta^2 \end{cases}, \qquad j = 3, \dots, n.$$

(1/20) (a) If $\mathbf{c} : [a, b] \to S$ is a curve parameterized by arclength then we define its **normal curvature in the direction of $\mathbf{e}_j$** as

$$k^{(j)}(s) = \mathbf{c}''(s) \cdot \mathbf{e}_j.$$

Show that if

$$\mathbf{c}'(s) = V^1(s)\mathbf{e}_1 + V^2(s)\mathbf{e}_2$$

then

$$k^{(j)} = \sum_{k,l=1}^{2} b_{kl}^{(j)} V^k V^l.$$

Conclude that if $k_1^{(j)}$ and $k_2^{(j)}$ are defined at a given point in S as the maximum and the minimum values of $k^{(j)}$ over all curves on S at that point then

$$k_1^{(j)} k_2^{(j)} = b_{11}^{(j)} b_{22}^{(j)} - b_{12}^{(j)} b_{21}^{(j)}.$$

(1/20) (b) Show that ${\omega_i}^j = -{\omega_j}^i$.

(1/20) (c) Starting with $d^2\mathbf{g} = 0$, prove that

$$\begin{cases} d\theta^1 = \theta^2 \wedge {\omega_2}^1 \\ d\theta^2 = \theta^1 \wedge {\omega_1}^2 \end{cases}.$$

(1/20) (d) Show that $b_{12}^{(j)} = b_{21}^{(j)}$.

(1/20) (e) Starting with $d^2\mathbf{e}_i = 0$, prove that $d{\omega_i}^j = \sum_{k=1}^{n} {\omega_i}^k \wedge {\omega_k}^j$.

(1/20) (f) Show that the Gauss curvature of S is given by $K = \sum_{j=3}^{n} k_1^{(j)} k_2^{(j)}$.