Differential Geometry of Curves and Surfaces

Homework 6

Due on October 27

1. Consider the 3-dimensional manifold with boundary
$$M = \{(x, y, z, w) \in \mathbb{R}^4 : x^2 + y^2 + z^2 + w^2 = 2 \wedge x^2 + y^2 \leq 1\}$$
and the 2-form
$$\omega = (-ydx + xdy) \wedge (-zdw + wdz) \in \Omega^2(\mathbb{R}^4).$$
Compute
$$\int_M d\omega$$
for your favorite choice of orientation of M by using:
   (a) The definition of integral on M.
   (b) The Stokes Theorem.

2. Consider the compact 1-dimensional manifold
$$S^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\},$$
oriented in the anti-clockwise direction, and the 1-form
$$\omega = -\frac{y}{x^2 + y^2}dx + \frac{x}{x^2 + y^2}dy \in \Omega^1(\mathbb{R}^2 \setminus \{(0, 0)\}).$$
Show that:
   (a) $d\omega = 0$.
   (b) $\displaystyle\int_{S^1} \omega = 2\pi$.
   (c) There does not exist a smooth map $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$ whose image is contained in S^1 and whose restriction to S^1 is the identity. (**Hint:** Consider the 1-form $\mathbf{f}^*\omega \in \Omega^1(\mathbb{R}^2)$)