Differential Geometry of Curves and Surfaces

Homework 5

Due on October 20

1. Consider the following differential forms:

$$\begin{aligned}
&\alpha = xdx + ydy \in \Omega^1(\mathbb{R}^2);\\
&\beta = -\frac{y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy \in \Omega^1(\mathbb{R}^2 \setminus \{\mathbf{0}\});\\
&\omega = e^{xz}dx + x\cos z dy + y^2 dz \in \Omega^1(\mathbb{R}^3);\\
&\eta = xdx \wedge dy - zdx \wedge dz + xyzdy \wedge dz \in \Omega^2(\mathbb{R}^3).
\end{aligned}$$

Consider also the following smooth functions:

$\mathbf{f} : \mathbb{R} \to \mathbb{R}^2$ defined as $\mathbf{f}(t) = (t, t^2)$;

$\mathbf{g} : (0, +\infty) \times (0, 2\pi) \to \mathbb{R}^2$ defined as $\mathbf{g}(r, \theta) = (r\cos\theta, r\operatorname{sen}\theta)$;

$\mathbf{h} : \mathbb{R}^3 \to \mathbb{R}^3$ defined as $\mathbf{h}(u, v, w) = (uv, vw, uw)$.

Compute:

(a) $\alpha \wedge \beta$, $\omega \wedge \eta$, $\eta \wedge \eta$;

(b) $d\alpha, d\beta, d\omega, d\eta$;

(c) $\mathbf{f}^*\alpha, \mathbf{g}^*\alpha, \mathbf{g}^*\beta, \mathbf{h}^*\eta$.

2. Recall that for any $\mathbf{v} \in \mathbb{R}^3$ we define

$$\omega_{\mathbf{v}} = v^1 dx + v^2 dy + v^3 dz$$

and

$$\Omega_{\mathbf{v}} = v^1 dy \wedge dz + v^2 dz \wedge dx + v^3 dx \wedge dy.$$

Show that if $\phi : \mathbb{R}^3 \to \mathbb{R}$ is a scalar field and $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ is a vector field then:

(a) $d\phi = \omega_{\operatorname{grad}\phi}$;

(b) $d\omega_{\mathbf{F}} = \Omega_{\operatorname{curl}\mathbf{F}}$;

(c) $d\Omega_{\mathbf{F}} = (\operatorname{div}\mathbf{F})\, dx \wedge dy \wedge dz$;

(d) $d(d\phi) = 0 \Leftrightarrow \operatorname{curl}(\operatorname{grad}\phi) = \mathbf{0}$;

(e) $d(d\omega_{\mathbf{F}}) = 0 \Leftrightarrow \operatorname{div}(\operatorname{curl}\mathbf{F}) = 0$.