Differential Geometry of Curves and Surfaces

Homework 11

Due on December 15

1. A **geodesic triangle** on a Riemannian surface is a simply connected domain with corners whose boundary is formed by three geodesic segments. Let α, β and γ be the interior angles at each corner. Show that:

   (a) If $K > 0$ then $\alpha + \beta + \gamma > \pi$.

   (b) If $K < 0$ then $\alpha + \beta + \gamma < \pi$.

   (c) The area of a geodesic triangle in the hyperbolic plane is always smaller than π.

2. Let A be a simply connected domain on a Riemannian surface with metric

$$ds^2 = \left(\theta^1\right)^2 + \left(\theta^2\right)^2,$$

where $\{\theta^1, \theta^2\}$ is dual to a positive orthonormal frame. Suppose that $\mathbf{V}(s)$ is a unit vector field parallel along a parametrization $\mathbf{c} : [s_0, s_1] \to \partial A$ by arclength in the positive direction. Show that if $\varphi(s)$ is the angle between $\mathbf{c}'(s)$ and $\mathbf{V}(s)$, given by $\cos(\varphi(s)) = \langle \mathbf{c}'(s), \mathbf{V}(s) \rangle$, then

$$\varphi(s_1) - \varphi(s_0) = \int_A K\theta^1 \wedge \theta^2 \mod 2\pi,$$

where K is the Gauss curvature. In other words, parallel transport along a closed loop rotates vectors by an angle equal to the integral of the Gauss curvature on the domain bounded by the loop.

Hint: Recall that if $\mathbf{V}(s)$ and $\mathbf{W}(s)$ are vector fields along $\mathbf{c}(s)$ then

$$\frac{d}{ds} \langle \mathbf{V}(s), \mathbf{W}(s) \rangle = \left\langle \frac{D\mathbf{V}}{ds}(s), \mathbf{W}(s) \right\rangle + \left\langle \mathbf{V}(s), \frac{D\mathbf{W}}{ds}(s) \right\rangle.$$