Differential Geometry of Curves and Surfaces

Homework 10

Due on December 7

1. Recall that the **hyperbolic plane** is the open domain $H = \{(x, y) \in \mathbb{R}^2 : y > 0\}$ with the Riemannian metric
$$ds^2 = \frac{1}{y^2}\left(dx^2 + dy^2\right).$$

   Show that:

   (a) The geodesic equations can be written as
$$\begin{cases} \ddot{x} - \dfrac{2}{y}\dot{x}\dot{y} = 0 \\ \\ \ddot{y} + \dfrac{1}{y}\dot{x}^2 - \dfrac{1}{y}\dot{y}^2 = 0 \end{cases}$$

   (b) The quantities
$$E = \frac{1}{y^2}\left(\dot{x}^2 + \dot{y}^2\right), \qquad p = \frac{\dot{x}}{y^2} \qquad \text{and} \qquad q = \frac{x\dot{x}}{y^2} + \frac{\dot{y}}{y}$$
   are constant along any geodesic.

   (c) When $p = 0$ then the geodesic is given by
$$(x(t), y(t)) = \left(x_0, y_0 e^{\pm\sqrt{E}t}\right),$$
   where x_0 and $y_0 > 0$ are constants.

   (d) When $p \neq 0$ then the image of the geodesic is contained in the circle
$$(px - q)^2 + p^2y^2 = E$$
   (with center in the x-axis).