Differential Geometry of Curves and Surfaces

Homework 1

Due on September 22

1. Consider the regular closed plane curve $\mathbf{c} : [0, 2\pi] \to \mathbb{R}^2$ given by

$$\mathbf{c}(t) = (r(t)\cos(t), r(t)\sin(t)),$$

where $r : [0, 2\pi] \to \mathbb{R}^+$ is a smooth function such that $r^{(n)}(0) = r^{(n)}(2\pi)$ for all $n \in \mathbb{N}$.

   (a) Show that the curvature of this curve is

$$k(t) = \frac{r^2 + 2\dot{r}^2 - r\ddot{r}}{(r^2 + \dot{r}^2)^{\frac{3}{2}}}.$$

   (b) If $r(t)$ has minimum $r(t_m)$ and maximum $r(t_M)$, show that

$$k(t_m) \leq \frac{1}{r(t_m)} \quad \text{and} \quad k(t_M) \geq \frac{1}{r(t_M)}.$$

Use a symbolic computation system, such as Mathematica, to solve the next problem

2. Consider the regular closed plane curve $\mathbf{c} : [0, 2\pi] \to \mathbb{R}^2$ given by

$$\mathbf{c}(t) = (3\cos(t) + 2\cos(2t), 3\sin(t) + 2\sin(2t)).$$

   (a) Plot the curvature of this curve as well as its derivative. Based on these plots, how many vertices does the curve have?

   (b) Plot the curve itself and explain why it does not violate the (general version of the) Four Vertex Theorem.