Differential Geometry of Curves and Surfaces
2023/2024

2nd Exam - 29 January 2024 - 9:00
Duration: 2 hours

(2/20) **1.** Let $\mathbf{c} : [0, 2\pi] \to \mathbb{R}^2$ be a convex curve parametrized by the parameter t such that its unit tangent vector is

$$\mathbf{e}_1(t) = (\cos t, \sin t)$$

(note that t is well defined because the curve is convex). If we set, as usual,

$$\mathbf{e}_2(t) = (-\sin t, \cos t),$$

the the **width** of the curve at $\mathbf{c}(t)$ is

$$W(t) = (\mathbf{c}(t+\pi) - \mathbf{c}(t)) \cdot \mathbf{e}_2(t) = -\mathbf{c}(t+\pi) \cdot \mathbf{e}_2(t+\pi) - \mathbf{c}(t) \cdot \mathbf{e}_2(t).$$

Prove that the length of the curve is

$$L = \int_0^\pi W(t)dt.$$

2. Consider the hyperboloid

$$S = \left\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 - z^2 = 1\right\}.$$

(1/20) (a) Prove that S is a manifold and find its dimension.

(1/20) (b) Determine $T_{(1,1,1)}S$.

(2/20) (c) Compute $\displaystyle\int_{S \cap \{0<z<1\}} d(xzdy - yzdx)$ for your choice of orientation.

(2/20) (d) Show that the intersection of S with the planes $x = 1$ and $z = 0$ is the union of the images of finitely many geodesics.

(e) Compute the mean curvature H and the Gauss curvature K of S using the parameterization $\mathbf{g} : \mathbb{R} \times (0, 2\pi) \to S$ given by

$$\mathbf{g}(u, \varphi) = (\cosh u \cos\varphi, \cosh u \sin\varphi, \sinh u)$$

by resorting to:

(2/20) (i) The classical formulas.

(2/20) (ii) The method of orthonormal frames.

(2/20) (f) Show that the angle by which a vector is rotated as it is parallel-transported once around $S \cap \{z = 1\}$ is given by $\int_{S \cap \{0<z<1\}} K$.

(2/20) **3.** A **ruled surface** in any surface admitting a parameterization $\mathbf{g} : \mathbb{R}^2 \to \mathbb{R}^3$ given by

$$\mathbf{g}(u, v) = \mathbf{c}(u) + v\mathbf{w}(u),$$

where $\mathbf{c} : \mathbb{R} \to \mathbb{R}^3$ is a curve and $\mathbf{w} : \mathbb{R} \to \mathbb{R}^3 \setminus \{\mathbf{0}\}$ is a nonvanishing vector. Prove that the Gauss curvature of a ruled surface vanishes if and only

$$\dot{\mathbf{c}}(u) \cdot (\mathbf{w}(u) \times \dot{\mathbf{w}}(u)) = 0$$

for all $u \in \mathbb{R}$. Given $\mathbf{c}(u)$, give three possible choices of functions $\mathbf{w}(u)$ that satisfy this condition (the resulting surfaces are called **developable surfaces**).

(2/20) **4.** The **third fundamental form** of a surface with parameterization $\mathbf{g}(u, v)$ and unit normal vector $\mathbf{n}(u, v)$ is defined as $\mathbf{III} = d\mathbf{n} \cdot d\mathbf{n}$. Prove that $K\mathbf{I} - 2H\mathbf{II} + \mathbf{III} = 0$.

Hint: Use the method of moving frames.

(2/20) **5.** Identify the minimal surface with Weierstrass-Enneper data $f(w) = e^{iw}$ and $g(w) = e^{-iw}$.