Differential Geometry of Curves and Surfaces

2023/2024

1^{st} Exam - 15 January 2024 - 9:00
Duration: 2 hours

(2/20) **1.** Let $\gamma : [a,b] \to \mathbb{C}$ be a closed regular curve, where we use the standard identification $\mathbb{R}^2 \simeq \mathbb{C}$. Show that the rotation index of γ is given by

$$m = \frac{1}{2\pi} \operatorname{Im} \int_a^b \frac{\ddot{\gamma}(t)}{\dot{\gamma}(t)} dt.$$

Moreover, show that if $f : \mathbb{C} \to \mathbb{C}$ is a holomorphic function whose derivative does not vanish on the image of γ then the rotation index of the curve $\Gamma = f \circ \gamma$ is

$$m' = m + \frac{1}{2\pi} \operatorname{Im} \int_\Gamma \frac{f''(z)}{f'(z)} dz.$$

Hint: Start by showing that the curvature of γ is $k(t) = \dfrac{\operatorname{Im}(\overline{\dot{\gamma}(t)}\ddot{\gamma}(t))}{|\dot{\gamma}(t)|^3}$.

2. Consider the differential forms defined on $\mathbb{R}^3 \setminus \{x = y = 0\}$ by

$$\omega = -\frac{y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy \qquad \text{and} \qquad \eta = \omega \wedge dz.$$

Show that:

(2/20) (a) ω is closed but not exact.

(2/20) (b) η is exact.

3. Consider the hyperbolic paraboloid

$$P = \left\{(x,y,z) \in \mathbb{R}^3 : z = xy\right\}.$$

(2/20) (a) Compute the Gauss curvature and the mean curvature of P.

(1/20) (b) Show that the curves of constant x are (images of) geodesics.

(1/20) (c) Prove that if a surface $S \subset \mathbb{R}^3$ contains a straight line then its Gauss curvature cannot be positive on the points of that line.

4. Consider the Riemannian metric

$$ds^2 = \frac{1}{\cos^2 y}\left(dx^2 + dy^2\right)$$

defined on the open set $\mathbb{R} \times \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

(2/20) (a) Show that the lines of constant x are (images of) geodesics.

(1/20) (b) Compute the Gauss curvature of this metric.

(1/20) (c) Prove that two geodesics with different images can intersect at most in one point.

(2/20) (d) Determine the vector field $\mathbf{V}(t)$, parallel along the curve $\mathbf{c}(t) = (t, y_0)$ (for some choice of $0 < y_0 < \frac{\pi}{2}$), satisfying $\mathbf{V}(0) = \frac{\partial}{\partial y}$. Is the curve $\mathbf{c}(t)$ turning left or right?

(2/20) **5.** Suppose that the compact surface $S \subset \mathbb{R}^3$ is the boundary of a bounded open set $A \subset \mathbb{R}^3$. Given a parameterization $\mathbf{g} : U \subset \mathbb{R}^2 \to S$ of S, consider the small deformation

$$\mathbf{g}_\varepsilon(u, v) = \mathbf{g}(u, v) + \varepsilon f(u, v)\mathbf{n}(u, v),$$

where $f : U \to \mathbb{R}$ has compact support and $\mathbf{n} : U \to S^2$ is the unit normal pointing away from A. If $V(\varepsilon)$ is the volume of the open set A_ε whose boundary is the surface S_ε parameterized by $\mathbf{g}_\varepsilon$, show that

$$\frac{dV}{d\varepsilon}(0) = \iint_U f\sqrt{EG - F^2}\,dudv.$$

Assuming that the method of Lagrange multipliers still works in this context, prove that of all surfaces that bound open sets with a given fixed volume, the one with minimal area (if it exists) must have constant mean curvature.

(2/20) **6.** Use the Weierstrass-Enneper representation to show that if a minimal surface is flat then it is contained in a plane.