Differential Geometry of Curves and Surfaces

2023/2024

Mock Exam
Duration: 2 hours

(2/20) **1.** Compute the maximum and the minimum curvatures of the ellipse

$$E = \left\{ (x, y) \in \mathbb{R}^2 : \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \right\},$$

where $a > b > 0$, using the parameterization $\mathbf{g} : [0, 2\pi] \to E$ given by

$$\mathbf{g}(t) = (a \cos t, b \sin t).$$

2. Consider the set

$$M = \{(x, y, z, u, v) \in \mathbb{R}^5 : x^2 + y^2 + z^2 = 2, u^2 + v^2 = 1, z > 0\}.$$

(1/20) (a) Show that M is a differentiable manifold and find its dimension.

(1/20) (b) Determine the tangent space to M at the point $(0, 1, 1, 1, 0)$.

(2/20) (c) Compute $\int_M d\omega$, where $\omega = xudy \wedge dv$, for your choice of orientation.

3. Compute the mean and the Gauss curvatures of the ellipsoid

$$S = \left\{ (x, y, z) \in \mathbb{R}^3 : \frac{x^2}{a^2} + \frac{y^2}{a^2} + \frac{z^2}{b^2} = 1 \right\},$$

where $a, b > 0$, by using the parameterization $\mathbf{g} : (0, \pi) \times (0, 2\pi) \to E$ given by

$$\mathbf{g}(\theta, \varphi) = (a \sin\theta \cos\varphi, a \sin\theta \sin\varphi, b \cos\theta),$$

and resorting to:

(2/20) (a) The classical formulas.

(2/20) (b) The method of orthonormal frames.

4. Consider $\mathbb{R}^2$ endowed with the Riemannian metric

$$ds^2 = \frac{1}{\cosh^2 y}\left(dx^2 + dy^2\right).$$

(2/20) (a) Show that the lines of constant x are (images of) geodesics. What can you say about the lines of constant y?

(1/20) (b) Prove that the inner product between the unit tangent vector to any geodesic and the vector field $\frac{\partial}{\partial x}$ is constant.

(1/20) (c) Compute the Gauss curvature of this metric.

(2/20) (d) Use the Gauss-Bonnet Theorem to find $\lim_{b\to\infty} \text{Area}(\Delta(a,b))$, where

$$\Delta(a,b) = \{(x,y) \in \mathbb{R}^2 : 0 < x < a \text{ and } 0 < y < b\}.$$

(2/20) **5.** A compact orientable surface S can be decomposed into finitely many pentagons (that is, images by some parameterization of Euclidean pentagons) whose intersections are precisely a common edge, such that exactly three edges meet at each vertex. Which surface is S, and how many pentagons are there in the decomposition?

(2/20) **6.** Let $S \subset \mathbb{R}^3$ be a surface with Gauss curvature $K = -1$. Prove that S is not a minimal surface.