Partial Differential Equations

Homework 9

Due on May 8

1. Let $f \in \mathcal{S}(\mathbb{R})$ satisfy $\|f\|_{L^2(\mathbb{R})} = 1$ (so that both $|f(x)|^2$ and $|\widehat{f}(\xi)|^2$ are probability density functions).

   (a) Using Plancherel's theorem, show that
   $$\int_{\mathbb{R}} |f'(x)|^2 \, dx = 4\pi^2 \int_{\mathbb{R}} \xi^2 |\widehat{f}(\xi)|^2 \, d\xi.$$

   (b) Integrate by parts to conclude that
   $$\mathrm{Re} \int_{\mathbb{R}} x\, f(x)\, \overline{f'(x)} \, dx = -\frac{1}{2}.$$

   (c) Apply the Cauchy–Schwarz inequality to deduce **Heisenberg's uncertainty principle**:
   $$\left(\int_{\mathbb{R}} x^2 |f(x)|^2 \, dx \right) \left(\int_{\mathbb{R}} \xi^2 |\widehat{f}(\xi)|^2 \, d\xi \right) \geq \frac{1}{16\pi^2}.$$

2. Consider the initial value problem for the wave equation in $\mathbb{R}$:
   $$\begin{cases} u_{tt} - u_{xx} = 0, \ x \in \mathbb{R}, \ t > 0, \\ u(x,0) = g(x), \ u_t(x,0) = h(s). \end{cases}$$

   (a) Arguing formally, show that the Fourier transform $\widehat{u}(\xi, t)$ of $u(x,t)$ with respect to the spatial variable x must satisfy
   $$\widehat{u}(\xi, t) = \widehat{g}(\xi) \cos(2\pi\xi t) + \widehat{h}(\xi) \frac{\sin(2\pi\xi t)}{2\pi\xi}.$$

   (b) Show that
   $$\mathcal{F}\left(\delta_t + \delta_{-t}\right)(\xi) = 2\cos(2\pi\xi t).$$

   (c) Assuming that the usual properties of the Fourier transform on $\mathcal{S}(\mathbb{R})$ relating products, convolutions and translations appropriately extend to tempered distributions, check that the first term in the expression of $\widehat{u}(\xi, t)$ is consistent with d'Alembert's formula.