Partial Differential Equations

Homework 7

Due on April 24

1. Show that if $g \in C^2(\mathbb{R})$ then the problem

$$\begin{cases} u_{tt} - u_{xx} = 0, & t > |x|, \\ u(x,x) = g(x), & x \geq 0, \\ u(x,-x) = g(x), & x \leq 0 \end{cases}$$

has a unique solution $u \in C^2(\overline{D})$, where

$$D = \left\{(x,t) \in \mathbb{R}^2 : t > |x|\right\},$$

and write it explicitly.

Remark: In particular, it is possible to give (half the usual) initial data for the wave equation on an appropriate union of intersecting characteristic surfaces. Is the solution that you found still unique in the region $t < |x|$? Would the analogue problem have a unique solution in the region $t > x$?

2. Consider the initial boundary value problem

$$\begin{cases} u_{tt} - u_{xx} = 0, & x \in (0,1), \quad t > 0, \\ u(x,0) = g(x), & x \in [0,1], \\ u_t(x,0) = h(x), & x \in [0,1], \\ u(0,t) = u(1,t) = 0, & t > 0, \end{cases}$$

where $g \in C^2([0,1])$ and $h \in C^1([0,1])$ satisfy

$$g(0) = g''(0) = h(0) = g(1) = g''(1) = h(1) = 0.$$

Show that if g and h are extended to $[-1,1]$ as odd functions, and then extended to $\mathbb{R}$ as periodic functions with period 2, then the D'Alembert formula gives a (unique) solution $u \in C^2(\overline{D})$ to this problem, where

$$D = \{(x,t) \in \mathbb{R}^2 : x \in (0,1), t > 0\}.$$