Partial Differential Equations

Homework 6

Due on April 2

1. Consider the following initial value problem for the heat equation:

$$\begin{cases} u_t - \Delta u = 0, & x \in \mathbb{R}^N, t > 0, \\ u(x,0) = g(x), & x \in \mathbb{R}^N, \end{cases}$$

   where $g \in C_c(\mathbb{R}^N)$. Show that the unique solution $u(x,t)$ of this problem in the Tychonov class satisfies:

   (a) $\displaystyle\int_{\mathbb{R}^n} u(x,t)dx = \int_{\mathbb{R}^n} g(x)dx.$

   (b) $\displaystyle\int_{\mathbb{R}^n} xu(x,t)dx = \int_{\mathbb{R}^n} xg(x)dx.$

   (c) $\displaystyle\sup_{(x,t)\in\mathbb{R}^N\times[0,+\infty)} u(x,t) = \sup_{x\in\mathbb{R}^N} g(x).$

   Remark: In particular, if g is a probability density function then $u(\cdot,t)$ is also a probability density function with the same average for each $t > 0$. Can you compute its variance?