Partial Differential Equations

Homework 5

Due on March 27

1. Consider the following initial value problem for the Laplace equation:

$$\begin{cases} u_{xx} + u_{yy} = 0 \\ u(x,0) = 0 \\ u_y(x,0) = g_n(x) \end{cases},$$

   where

$$g_n(x) = \frac{1}{n}\sin(nx).$$

   Show that:

   (a) $u_n(x,y) = \frac{1}{n^2}\sin(nx)\sinh(ny)$ is a solution of the problem.

   (b) $\|g_n\|_{L^\infty(\mathbb{R})} \to 0$ as $n \to \infty$.

   (c) $\|u_n\|_{L^\infty(\mathbb{R}\times[0,\varepsilon])} \to \infty$ as $n \to \infty$ for any $\varepsilon > 0$.

   (d) How do the results above change if we replace the Lapace equation with the wave equation $u_{xx} - u_{yy} = 0$?

   Remark: This exercise shows why solving the Laplace equation as an initial value problem is a terrible idea.

2. Let $u \in C^2(\mathbb{R}^N)$ be a harmonic function such that $|u(x)| \le C|x|^k$ for $C > 0$ and $k \in \mathbb{N}_0$. Prove that u is a polynomial of degree k.