Partial Differential Equations

Homework 4

Due on March 20

1. Consider the linear second order equation in $\mathbb{R}^N$

$$\sum_{i,j=1}^{N} a_{ij}(x)u_{x_i x_j} + \sum_{i=1}^{N} b_i(x)u_{x_i} + c(x)u = d(x).$$

and let $\phi \in C^2(\mathbb{R}^N)$ be a function satisfying $\nabla\phi(x) \neq 0$ for all $x \in \mathbb{R}^N$.

(a) Show that if $y = y(x)$ is a coordinate transformation of class C^2 such that $y_1 = \phi$ and the functions $u, v \in C^2(\mathbb{R}^N)$ are such that $u(x) = v(y(x))$ then u satisfies the linear PDE above if and only if v satisfies the linear PDE

$$\sum_{i,j=1}^{N} \tilde{a}_{ij}(y)v_{y_i y_j} + \sum_{i=1}^{N} \tilde{b}_i(y)v_{y_i} + \tilde{c}(x)u = \tilde{d}(x),$$

where

$$\tilde{a}_{ij}(y(x)) = \sum_{k,l=1}^{N} \frac{\partial y_i}{\partial x_k}(x)\frac{\partial y_j}{\partial x_l}(x)a_{kl}(x),$$

$$\tilde{b}_i(y(x)) = \sum_{k,l=1}^{N} \frac{\partial^2 y_i}{\partial x_k \partial x_l}(x)a_{kl}(x) + \sum_{k=1}^{N} \frac{\partial y_i}{\partial x_k}(x)b_k(x),$$

$$\tilde{c}(y(x)) = c(x), \qquad \tilde{d}(y(x)) = d(x).$$

(b) Show that $\tilde{a}_{11} = 0$ on the hypersurface $\phi = 0$ if and only if

$$\sum_{i,j=1}^{N} a_{ij}\phi_{x_i}\phi_{x_j} = 0$$

on this hypersurface (which is then called **characteristic**, since it is impossible to recover $v_{y_1 y_1}$ from the equation and initial data for $(v, \nabla v)$ given on this hypersurface).

(c) Find characteristic surfaces for the following linear PDEs in $\mathbb{R}^3$:

(i) $u_t = u_{xx} + u_{yy}$.

(ii) $u_{tt} = u_{xx} + u_{yy}$.