Partial Differential Equations

Homework 13

Due on June 5

1. Let $\Omega \subset \mathbb{R}^N$ be a bounded open set. An **eigenfunction** of (minus) the Laplacian operator with Dirichlet boundary conditions and **eigenvalue** $\lambda \in \mathbb{R}$ is a nonvanishing weak solution of the problem
$$\begin{cases} -\Delta u = \lambda u \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega \end{cases} .$$
   (a) Prove that the eigenvalues of the Laplacian are necessarily positive.
   (b) Does the Lax-Milgram theorem forbid the existence of positive eigenvalues?

2. Let $\Omega \subset \mathbb{R}^N$ be a bounded open set and let $A : \overline{\Omega} \to M_{N\times N}(\mathbb{R})$ be a matrix of continuous functions such that $A(x)$ is symmetric and positive definite for all $x \in \overline{\Omega}$. Prove for any $f \in H^{-1}(\Omega)$ the problem
$$\begin{cases} -\operatorname{div}(A(x)\nabla u) = f \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega \end{cases}$$
has a unique weak solution.