Partial Differential Equations

Homework 12

Due on May 29

1. Consider the open set
$$\Omega = \{(x,y) \in \mathbb{R}^2 : |x| + |y| < 1\},$$
and let $f \in \mathcal{D}'(\Omega)$ be given by
$$\langle f, \varphi \rangle = \int_{-1}^{1} \varphi(x,0)dx + \int_{-1}^{1} \varphi(0,y)dy$$
for all $\varphi \in \mathcal{D}(\Omega)$.

   (a) Show that $f \in H^{-1}(\Omega)$.

   (b) Prove that the function
   $$u(x,y) = \frac{1}{2}(1 - |x| - |y|)$$
   is the unique weak solution of the problem
   $$\begin{cases} -\Delta u = f \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega \end{cases},$$
   that is, the unique function $u \in H_0^1(\Omega)$ such that
   $$\langle u, v \rangle_{H_0^1 \times H_0^1} = \langle f, v \rangle_{H^{-1} \times H_0^1}$$
   for all $v \in H_0^1(\Omega)$, where
   $$\langle u, v \rangle_{H_0^1 \times H_0^1} = \int_\Omega \nabla u \cdot \nabla v.$$