Partial Differential Equations
IST 2022/2023

Extra problem sheet - Fourier Transform

In this extra sheet, functions are complex valued: $u : \mathbb{R}^N \to \mathbb{C}$. Recall that we have defined the Fourier transform and the inverse Fourier transform of u as

$$\widehat{u}(\xi) = \int_{\mathbb{R}^N} e^{-2\pi i x\cdot\xi} u(x)\, dx, \qquad \breve{u}(x) = \int_{\mathbb{R}^N} e^{2\pi i x\cdot\xi} u(\xi)\, d\xi.$$

These operations are well defined, for instance, for $u \in L^1(\mathbb{R}^N)$; then $\widehat{u} \in L^\infty(\mathbb{R}^N) \cap C(\mathbb{R}^N)$ and $\|\widehat{u}\|_{L^\infty(\mathbb{R}^N)} \leqslant \|u\|_{L^1(\mathbb{R}^N)}$.

Under certain regularity assumptions on u, one can show that these operations are, indeed, the inverse of one another:

$$u(x) = \int_{\mathbb{R}^N} e^{2\pi i x\cdot\xi}\, \widehat{u}(\xi)\, d\xi \quad \text{and} \quad u(\xi) = \int_{\mathbb{R}^N} e^{-2\pi i x\cdot\xi}\, \breve{u}(x)\, dx.$$

Moreover,

$$\widehat{\frac{\partial u}{\partial x_j}}(\xi) = 2\pi i \xi_j \widehat{u}(\xi) \quad \text{and} \quad \frac{\partial}{\partial \xi_j}\widehat{u}(\xi) = -2\pi i (\widehat{x_j u})(\xi)$$

1. Algebraic properties of the Fourier Transform:
   (a) (linearity) The Fourier transform is linear, *i.e.*, the map $L^1(\mathbb{R}^N) \to L^\infty(\mathbb{R}^N)$, $u \mapsto \widehat{u}$ is linear;
   (b) (translation) $\widehat{u(\cdot - h)}(\xi) = e^{-2\pi i h\cdot\xi}\hat{u}(\xi)$ for $h \in \mathbb{R}^N$ and $u \in L^1(\mathbb{R}^N)$;
   (c) (dilation) $\widehat{u(\lambda \cdot)}(\xi) = \frac{1}{\lambda^N}\widehat{u}(\frac{\xi}{\lambda})$ for every $\lambda > 0$ and $u \in L^1(\mathbb{R}^N)$;
   (d) (convolution) Recalling that $u * v \in L^1(\mathbb{R}^N)$ for $u, v \in L^1(\mathbb{R}^N)$, show that $\widehat{u * v} = \widehat{u} \cdot \widehat{v}$.
2. For $u, v \in L^1(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$, show that
$$\int_{\mathbb{R}^N} \widehat{u} v = \int_{\mathbb{R}^N} u \widehat{v}.$$
3. Show that:
   (a) if u is radially symmetric, then $\widehat{u}$ is also radially symmetric.
   (b) if u is homogeneous of degree $\alpha \neq 0$, then $\widehat{u}$ is homogeneous of degree $-N - \alpha$.

In the next two exercises we are going to compute the Fourier Transform of two important classes of functions. The students who followed the *Real Analsysis course* should justify everything rigorously. The others should simply use the previous properties without worrying whether or not one can compute the Fourier transform of such functions.

4. We determine the Fourier transform of the Gaussian function:
$$\varphi(x) = e^{-\sigma|x|^2}, \quad \sigma > 0 \quad \Longrightarrow \quad \widehat{\varphi}(\xi) = \left(\frac{\pi}{\sigma}\right)^{\frac{N}{2}} e^{-\frac{\pi^2}{\sigma}|\xi|^2}$$
   (a) For $N = 1$, use the fact that $\varphi'(x) = -2\sigma x\varphi(x)$ in order to show that $v(\xi) = \widehat{\varphi}(\xi)$ is a solution to the initial value problem:
$$v'(\xi) = -\frac{2\pi^2}{\sigma}\xi v(\xi), \quad v(0) = \widehat{\varphi}(0).$$
   Conclude that $\widehat{\varphi}(\xi) = \widehat{\varphi}(0)e^{-\frac{\pi^2\xi^2}{\sigma}}$.

(b) Sill for $N = 1$, show that $\widehat{\varphi}(0) = \sqrt{\pi/\sigma}$. (recall the "famous" trick of using an auxiliary two variable function + polar coordinates). With this we conclude the desired result in dimension $N = 1$.

(c) Show the result for an arbitrary dimension N, using Fubini's Theorem.

5. We wish to compute the Fourier transform of $\varphi(x) = \frac{1}{|x|^\alpha}$, for $\alpha \in (0, N)$, by showing that:

$$\widehat{\varphi}(\xi) = \frac{c_{N,\alpha}}{|x|^{N-\alpha}}, \qquad \text{where} \qquad c_{N,\alpha} = \pi^{\alpha-\frac{N}{2}} \frac{\Gamma(N/2 - \alpha/2)}{\Gamma(\alpha/2)}.$$

Recall that the Gamma function is given by

$$\Gamma(t) = \int_0^\infty x^{t-1} e^{-x}\, dx, \qquad \text{and satisfies } \Gamma(1) = 1,\ \Gamma(t+1) = t\Gamma(t)\ (t > 0).$$

(a) Using exercise 3, show the existence of $c > 0$ such that

$$\widehat{\varphi}(\xi) = \frac{c}{|\xi|^{N-\alpha}}.$$

(b) To see that $c = c_{N,\alpha}$, reason in the following way:

i. Apply the identity of exercise 2 to $u = \varphi$, $v = e^{-\pi|x|^2}$.

ii. Use generalized polar coordinates to obtain:

$$\int_0^\infty x^{N-1-\alpha} e^{-\pi x^2}\, dx = c \int_0^\infty x^{\alpha-1} e^{-\pi x^2}\, dx$$

iii. Use a convenient change of variable and the definition of the Γ–function to conclude.

Final Remark. In terms of the Γ–function, the measure of the unitary sphere in $\mathbb{R}^N$ is given by:

$$|\partial B_1| = \frac{2\pi^{N/2}}{\Gamma(N/2)} \quad \left(= \frac{4\pi^{N/2}}{\Gamma(N/2-1)(N-2)} \right).$$

In order to show this, we write:

$$\int_{\mathbb{R}^N} e^{-|x|^2}\, dx = |\partial B_1| \int_0^\infty e^{-r^2} r^{N-1}\, dr$$

computing the integral in the left hand side and making a change of variable in the right hand side (try it out!). Therefore, for $N \geqslant 3$,

$$\widehat{\frac{1}{|\cdot|^{N-2}}}(\xi) = \frac{c_{N,N-2}}{|\xi|^2}, \qquad \text{where } c_{N,N-2} = \frac{\Gamma(1)\pi^{\frac{N}{2}-2}}{\Gamma(N/2-1)} = \frac{(N-2)|\partial B_1|}{4\pi^2}.$$

In particular,

$$\widehat{\Phi}(\xi) = \frac{1}{4\pi^2|\xi|^2},$$

where Φ is the fundamental solution of the Laplacian in dimension $N \geqslant 3$.