Partial Differential Equations
IST 2022/2023

Problem sheet 6 - Sobolev Spaces

1. Consider the function $u : B_1 \subset \mathbb{R}^N \to \mathbb{R}$ defined by

$$u(x, y) = |(x, y)|^\alpha,$$

where $B_1 = \{(x, y) \in \mathbb{R}^2 : \; x^2 + y^2 < 1\}$. For which $\alpha \in \mathbb{R}$ do we have $u \in H^1(B_1)$?

2. (Another proof of Poincaré's inequality in the special case $p = 2$). Let $\Omega \subset \mathbb{R}^N$ be an open set of $\mathbb{R}^N$.

   (a) Prove that, if $\Omega \subset \{x \in \mathbb{R}^N : \; -a < x_1 < a\}$ for some $a > 0$, then there exists $C = C(a) > 0$ such that

$$\|u\|_{L^2(\Omega)} \leqslant C \left\| \frac{\partial u}{\partial x_1} \right\|_{L^2(\Omega)} \qquad \forall u \in H^1_0(\Omega).$$

   (b) Suppose now that Ω is bounded. Use the previous step to show Poincaré's inequality: there exists $C > 0$ such that

$$\|u\|_{L^2(\Omega)} \leqslant C \, \|\nabla u\|_{L^2(\Omega)} \qquad \forall u \in H^1_0(\Omega).$$

3. (The Laplacian as an operator with values in $H^{-1}(\Omega)$). Let $\Omega \subset \mathbb{R}^N$ be an open set. Consider the map

$$\Delta : H^1_0(\Omega) \to \mathcal{D}'(\Omega), \quad u \mapsto \Delta u,$$

where Δu is the Laplacian of u in the sense of distributions.

   (a) Prove that, for each $u \in H^1_0(\Omega)$, we have $\Delta u \in H^{-1}(\Omega)$ and

$$\langle \Delta u, v \rangle_{H^{-1} \times H^1_0} = - \int_\Omega \nabla u \cdot \nabla v \, dx \qquad \forall u, v \in H^1_0(\Omega).$$

   (More precisely: we can extend $\Delta u \in \mathcal{D}'(\Omega)$ to an element in $H^{-1}(\Omega)$, and the previous formula holds true).

   (b) Show that the operator

$$\Delta : H^1_0(\Omega) \to H^{-1}(\Omega),$$

   is continuous.

4. (A general extension result) Let X be a normed space and $Y \subset X$ a dense subspace. Let Z be a Banach space and $A \in \mathcal{L}(Y, Z)$. Prove that there exits a unique extension $\bar{A} \in \mathcal{L}(X, Z)$ of A, and that $\bar{A}$ satisfies $\|\bar{A}\|_{\mathcal{L}(X,Z)} = \|A\|_{\mathcal{L}(Y,Z)}$.

5. (Show that $u \in W^{1,p}(\Omega) \implies |u| \in W^{1,p}(\Omega)$). Let Ω be a bounded domain of class C^1 and take $u \in W^{1,p}(\Omega)$ for some $p \in [1, +\infty)$.

   (a) Let $f \in C^1(\mathbb{R})$ be such that $f' \in L^\infty(\mathbb{R})$. Prove that:

      i. There exist $c_1, c_2 > 0$ such that $|f(t)| \leqslant c_1 + c_2|t|$ for every $t \in \mathbb{R}$, and so $f(u) \in L^p(\Omega)$.

      ii. In the sense of distributions,

$$\frac{\partial}{\partial x_i}(f(u)) = f'(u) \frac{\partial u}{\partial x_i} \qquad \text{for every } i = 1, \ldots, N.$$

      iii. Conclude that $f \circ u \in W^{1,p}(\Omega)$.

(b) Show that $|u| \in W^{1,p}(\Omega)$, with

$$\frac{\partial |u|}{\partial x_i}(x) = \begin{cases} \frac{\partial u}{\partial x_i}(x) & \text{if } u(x) > 0, \\ -\frac{\partial u}{\partial x_i}(x) & \text{if } u(x) < 0, \\ 0 & \text{otherwise.} \end{cases}$$

(Hint: Consider $f_\varepsilon(t) = \sqrt{t^2 + \varepsilon^2} - \varepsilon$ for $\varepsilon > 0$, apply item (a) to $f_\varepsilon(u)$ and then pass to the limit as $\varepsilon \to 0$).

(c) If $u \in W_0^{1,p}(\Omega) \cap C(\overline{\Omega})$, prove that $|u| \in W_0^{1,p}(\Omega) \cap C(\overline{\Omega})$.

6. Take $u \in W^{1,p}(\mathbb{R}^N)$, $1 \leqslant p < \infty$, and define the translation operator $\tau_h u(x) = u(x+h)$, $h \in \mathbb{R}^N$. Show that

$$\|\tau_h u - u\|_{L^p(\mathbb{R}^N)} \leqslant \|\nabla u\|_{L^p(\mathbb{R}^N)} |h|.$$