Partial Differential Equations
IST 2022/2023

Problem sheet 5 - Distributions

In what follows, Ω is always an open subset of $\mathbb{R}^N$, $N \geqslant 1$.
Recall also that, for any multi-index $\alpha = (\alpha_1, \dots, \alpha_N) \in \mathbb{N}_0^N$, we define $D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \dots \partial x_N^{\alpha_N}}$.

1. Show that, if $T_k \to T$ in $\mathcal{D}'(\Omega)$, then $D^\alpha T_k \to D^\alpha T$ in $\mathcal{D}'(\Omega)$, for every multi-index $\alpha \in \mathbb{N}_0^N$.
2. Given $T \in \mathcal{D}'(\Omega)$, show that $\operatorname{div}(\nabla T) = \Delta T$ in $\mathcal{D}'(\Omega)$.
3. Recall that the fundamental solution of the Laplacian in $\mathbb{R}^2$ is given by
$$\Phi(x) = -\frac{1}{2\pi} \log |x| = -\frac{1}{4\pi} \log(x_1^2 + x_2^2)$$
Prove that $-\Delta \Phi = \delta_0$ in $\mathcal{D}'(\mathbb{R}^2)$
4. Which of the following functionals $T : \mathcal{D}(\Omega) \to \mathbb{R}$ define distributions?
    (a) $\langle T, \phi \rangle = \sum_{k=1}^\infty k! \phi^{(k)}(k)$, for $\Omega = \mathbb{R}$.
    (b) $\langle T, \phi \rangle = \sum_{k=1}^\infty \frac{\phi(k)}{k}$, for $\Omega = \mathbb{R}$
    (c) $\langle T, \phi \rangle = \int_0^\infty \frac{\phi(x)}{x^2}\, dx$, for $\Omega = (0, \infty)$
5. (Approximations of δ_0):
    a) Let $B_r = B_r(0) \subset \mathbb{R}^N$. Show that
    $$\lim_{r \to 0} \frac{1}{|B_r|} \chi_{B_r} = \delta_0 \qquad \text{in } \mathcal{D}'(\mathbb{R}^N).$$
    b) Take $\eta \in C_c^\infty(\mathbb{R}^N)$ a nonnegative function such that $\operatorname{supp} \eta = \overline{B_1(0)}$ and $\int_{\mathbb{R}^N} \eta = 1$. Consider the *mollifier*:
    $$\eta_\varepsilon(x) = \frac{1}{\varepsilon^n} \eta\left(\frac{x}{\varepsilon}\right).$$
    Prove that $\lim_{\varepsilon \to 0} \eta_\varepsilon = \delta_0$ in $\mathcal{D}'(\mathbb{R}^N)$.
6. Let $T : \mathcal{D}(\Omega) \to \mathbb{R}$ be a linear functional. Show that $T \in \mathcal{D}'(\Omega)$ if, and only if, for each compact set K contained in Ω, there exist constants $C > 0$ and $m \in \mathbb{N}$ such that
$$|\langle T, \phi \rangle| \leqslant C \sum_{j=1}^m \sum_{|\alpha| = j} \|D^\alpha \phi\|_\infty, \qquad \text{for every } \phi \in D(K).$$
(Hint: Argue by contradiction to show the estimate.)
7. Compute the derivative in the sense of distributions of the following functions ($x \in \mathbb{R}$):
    a) $|x|$
    b) $\chi_{(a,b)}(x)$, for some $a < b$.
8. The purpose of this exercise is to show, in several steps, the following result.

    Let $-\infty \leqslant a \leqslant b \leqslant +\infty$, $\Omega =]a, b[$, and assume that $T \in \mathcal{D}'(\Omega)$ is such that $T' = 0$ (i.e., $\langle T, \varphi \rangle = 0$ for every $\varphi \in \mathcal{D}(\Omega)$). Then there exists $c \in \mathbb{R}$ such that $T = c$ in $\mathcal{D}'(\Omega)$.

(a) (warm up) Let $F \in \mathcal{D}'(\Omega)$ be defined by $\langle F, \varphi \rangle = \int_\Omega \varphi$ for every $\varphi \in \mathcal{D}(\Omega)$, and let $\ker F := \{\varphi \in \mathcal{D}(\Omega) : \langle F, \varphi \rangle = 0\}$.

i. Show the existence of $\varphi_0 \in \mathcal{D}(\Omega)$ such that $\langle F, \varphi_0 \rangle = 1$.

ii. Prove that $\mathcal{D}(\Omega) = \ker F + X$, where $X = \{\lambda \varphi_0 : \lambda \in \mathbb{R}\}$.

iii. For $\psi \in \ker F$, let

$$\Psi(x) := \int_a^x \psi(t)\, dt = \int_a^b \psi(t)\, dt - \int_x^b \psi(t)\, dt = -\int_x^b \psi(t)\, dt.$$

Show that $\Psi \in \mathcal{D}(\Omega)$ and that $\Psi' = \psi$ in $\mathcal{D}'(\Omega)$.

(b) (proof of the result) Now take $T \in \mathcal{D}'(\mathbb{R})$ such that $T' = 0$. Given any $\varphi \in \mathcal{D}(\Omega)$, write it as

$$\varphi = \psi + \lambda \varphi_0, \text{ for } \psi \in \ker F,\ \lambda \in \mathbb{R},\ \int_\Omega \varphi_0 = 1.$$

Show that $\langle T, \varphi \rangle = \langle c, \varphi \rangle$, for $c := \langle T, \varphi_0 \rangle$.