Partial Differential Equations
IST 2022/2023

Problem sheet 4 - The heat and wave equations

1. Let $u \in C^{2,1}(\mathbb{R}^N \times (0,\infty))$ be a solution of the heat equation $u_t - \Delta u = 0$ in $\mathbb{R}^N \times (0,\infty)$.

   (a) Let $v(x) = u(ax, bt)$, for $a, b \in \mathbb{R}$. Prove that v solves the heat equation if $b = a^2$.

   (b) If, moreover, $u \in C^3(\mathbb{R}^N \times (0,\infty))$, use (a) to show that $v(x,t) = \nabla u(x,t) \cdot x + 2tu_t(x,t)$ also solves the heat equation.

2. Let Ω be a bounded domain of $\mathbb{R}^N$ and let $u \in C^{2,1}(\overline{\Omega} \times (0,\infty)) \cap C(\overline{\Omega} \times [0,\infty))$ be a solution to the following initial value problem with Neumann boundary conditions

$$\begin{cases} u_t - \Delta u = 0, & x \in \Omega,\ t > 0 \\ u(x,0) = u_0(x), & x \in \Omega \\ \frac{\partial u}{\partial \nu}(x,t) = 0 & x \in \partial\Omega,\ t > 0. \end{cases}$$

   (a) Prove that the *total mass* of u is conserved, that is

$$\int_\Omega u(x,t)\,dx = \int_\Omega u_0(x)\,dx \qquad \text{for every } t > 0.$$

   Justify why this result is natural from a physical point of view.

   (b) Prove, using energy methods, that the problem has at most one solution.

3. Given $g \in C(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, consider the problem

$$\begin{cases} u_t - \Delta u = 0, & x \in \mathbb{R}^N,\ t > 0 \\ u(x,0) = g(x), & x \in \mathbb{R}^N. \end{cases}$$

   Prove that, if $g \in L^p(\mathbb{R}^N)$ for some $p \in [1,\infty)$, then $\sup_{x\in\mathbb{R}^N} |u(t,x)| \to 0$ as $t \to \infty$. Give a counterexample for the case $p = \infty$.

4. Let $g \in C^1([0,\infty)) \cap L^\infty(0,\infty)$ with $g(0) = 0$, and consider the heat equation in an half-line:

$$\begin{cases} u_t - u_{xx} = 0 & x, t \in (0,\infty) \\ u(x,0) = 0 & x \geqslant 0 \\ u(0,t) = g(t) & t \geqslant 0 \end{cases} \tag{1}$$

   Deduce, from the representation formula in $\mathbb{R}$, the following formula for solutions of (1)

$$u(x,t) = \frac{x}{\sqrt{4\pi}} \int_0^t \frac{1}{(t-s)^{3/2}} e^{-\frac{x^2}{4(t-s)}} g(s)ds.$$

   (Hint: Take $v(x,t) = u(x,t) - g(t)$ and extend v for $x < 0$ in an odd fashion).

5. Consider the functional

$$E(u,v) = \frac{1}{2}\int_\Omega (v(x))^2 + c^2(u_x(x))^2 dx, \quad (u,v) \in D(E) := (C^2(\overline{\Omega}) \cap C_0(\bar{\Omega})) \times C_0(\overline{\Omega}),$$

   where $C_0(\overline{\Omega}) = \{w \in C(\overline{\Omega}) : u = 0 \text{ in } \partial\Omega\}$ and Ω is an open subset of $\mathbb{R}$.

   One defines the derivative of E at (u,v) along the direction $(\varphi,\psi) \in D(E)$ as

$$E'(u,v)[\varphi,\psi] := \frac{d}{dt} E(u + t\varphi, v + t\psi)\Big|_{t=0}.$$

   The gradient $\nabla E(u,v)$ in $L^2(\Omega) \times L^2(\Omega)$ is defined through the identity $E'(u,v)[\varphi,\psi] = \langle \nabla E(u,v), (\varphi,\psi)\rangle_{L^2\times L}$ for every (φ,ψ).

(a) Show (formally) that

$$\nabla E(u,v) = \big(-c^2 u_{xx}, v\big)$$

(b) If $u \in C^2(\Omega \times [0,T])$ solves the wave equation in Ω with Dirichlet boundary conditions, check that, for $v := u_t$,

$$(u(t), v(t))_t = J \cdot \nabla E(u(t), v(t)), \quad J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

[Remark: A system of this form is called *Hamiltonian*.]

(c) Using the Hamiltonian formulation, prove that E is conserved along the solutions of the wave equation.

6. Let $a \in C_c^1(\mathbb{R})$ be a nonnegative function and let $f \in C^2(\mathbb{R})$, $g \in C^1(\mathbb{R})$. Consider the initial value problem for the wave-type equation:

$$\begin{cases} u_{tt} - (a(x)u_x)_x = 0, & x \in \mathbb{R}, t > 0, \\ u(x,0) = f(x),\ u_t(x,0) = g(x), & x \in \mathbb{R}. \end{cases} \tag{2}$$

Use the energy method to show that the problem (2) admits at most one solution $u \in C^2(\mathbb{R} \times [0,\infty))$.