Partial Differential Equations
IST 2022/2023

Problem sheet 3 - The Laplace and Poisson equations

In what follows, Ω is an open and connected subset of $\mathbb{R}^N$.

1. Consider a *hyperbolic* equation with *constant coefficients* of the form
$$au_{xx} + 2bu_{xy} + cu_{yy} + du_x + eu_y + fu = 0 \text{ in } \mathbb{R}^2.$$
Show that the change of variables
$$\xi = ay - (b + \sqrt{b^2 - ac})x, \quad \eta = ay - (b - \sqrt{b^2 - ac})x$$
reduces the equation to the canonical form.

2. Show that the classification of linear second order PDEs is invariant by change of variables (as it should be!). This means the following: if $u \in C^2(\Omega)$ solves
$$a(x,y)u_{xx} + 2b(x,y)u_{xy} + c(x,y)u_{yy} + d(x,y)u_x + e(x,y)u_y + \sigma(x,y)u = F(x,y), \tag{1}$$
and $\xi = \xi(x,y)$, $\eta = \eta(x,y)$ are such that
$$\begin{vmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{vmatrix} \neq 0,$$
then if (1) is elliptic/parabolic/hyperbolic in Ω, and v is defined through $v(\xi(x,y), \eta(x,y)) = u(x,y)$, then the equation satisfied by v is respectively elliptic/parabolic/hyperbolic.

3. Show that, if u is harmonic in $\mathbb{R}^N$ and O is an orthonormal $N \times N$ matrix, then $v(x) := u(Ox)$ is also harmonic in $\mathbb{R}^N$.

4. Let $u \in C^2(\Omega)$ be a subharmonic function in Ω, that is,
$$-\Delta u \leqslant 0 \text{ in } \Omega.$$
   (a) Show that, for every ball $B_r(x) \subset \Omega$, we have
$$u(x) \leqslant \frac{1}{|\partial B_r|} \int_{\partial B_r(x)} u(y)\, d\sigma(y) \quad \text{and} \quad u(x) \leqslant \frac{1}{|B_r|} \int_{B_r(x)} u(y)\, dy.$$
   (b) If, moreover, $u \in C^2(\Omega) \cap C(\overline{\Omega})$ is non constant, prove that
$$u(x) < \max_{\partial\Omega} u \quad \forall x \in \Omega.$$

5. (a) Give an example of a subharmonic function in $\mathbb{R}^N$ which is not harmonic.
   (b) Prove that, if u is harmonic in $\mathbb{R}^N$ and $F \in C^2(\mathbb{R})$ is convex, then $F(u)$ is subharmonic.

6. (a) Verify that the function
$$u(x,y) = \frac{1 - x^2 - y^2}{1 - 2x + x^2 + y^2}$$
is harmonic in $B_1(0,0) \subset \mathbb{R}^2$.
   (b) Since $1 - x^2 - y^2 = 0$ on $\partial B_1(0,0)$, should not we have $u \equiv 0$, by the maximum principle? Explain this (apparent) contradiction.
   (c) Compute the maximum of u in $\overline{B_{1/2}(0,0)}$.

7. Given $f \in C(\overline{\Omega})$ and $g \in C(\partial\Omega)$, consider the boundary value problem

$$-\Delta u = f \text{ in } \Omega, \qquad u = g \text{ on } \partial\Omega. \tag{2}$$

   (a) Prove there exists a constant C, depending only on Ω, such that for every solution $u \in C^2(\Omega) \cap C(\overline{\Omega})$ of (2),

   $$\max_{\overline{\Omega}} |u| \leqslant C(\max_{\partial\Omega} |g| + \max_{\overline{\Omega}} |f|)$$

   (Hint: Consider the auxiliary function $v(x) = u(x) + \frac{M}{2N}|x|^2$, where $M := \max_{\overline{\Omega}} |f|$.)

   (b) Based on (a), state and prove a stability result for solutions of the problem (2).

8. Prove the following properties:

   (a) If u and u^2 are harmonic in Ω, then u is constant.

   (b) If u is harmonic in $\mathbb{R}^N$ then $|\nabla u|^2$ is subharmonic.

   (c) Let $\Omega \subset \mathbb{R}^2$ be a bounded domain and $v \in C^3(\overline{\Omega})$ be a solution of

   $$-\Delta v = 1 \text{ in } \Omega, \qquad v = 0 \text{ on } \partial\Omega$$

   (v is sometimes called the *torsion* function). Show that $|\nabla v|^2$ attains its maximum on $\partial\Omega$.

9. Deduce the mean value formula for harmonic functions in $\mathbb{R}^3$ from Green's representation formula.

10. (a) Let $h \in C(\partial\Omega)$. Show that, whenever it exists a solution $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ of the *Neumann* problem

   $$\Delta u = 0 \text{ in } \Omega, \qquad \frac{\partial u}{\partial \nu} = h \text{ on } \partial\Omega,$$

   it is unique up to a constant (*i.e.*, if u_1 and u_2 are two solutions, then there exists $c \in \mathbb{R}$ such that $u_1 \equiv u_2 + c$). Hint: use energy methods.

   (b) Give a *necessary* condition on h for the existence of a solution for the Neumann problem.