Partial Differential Equations
IST 2022/2023

Problem sheet 2 - First order partial differential equations

1. Given $g \in C(\mathbb{R}^N)$ and $b \in \mathbb{R}^N$, determine the unique solution of the initial value problem

$$\begin{cases} u_t + b \cdot \nabla u + cu = 0 \text{ in } \mathbb{R}^N \times \mathbb{R} \\ u(x,0) = g(x) \text{ on } \mathbb{R}^N. \end{cases}$$

2. Given $q \in C^1(\mathbb{R})$ and $g \in C(\mathbb{R})$, consider the following scalar conservation law with unknown $u = u(x,t)$:

$$\begin{cases} u_t + (q(u))_x = 0, & x \in \mathbb{R}, t \geqslant 0, \\ u(x,0) = g(x), & x \in \mathbb{R}. \end{cases} \tag{1}$$

   (a) Prove that, if $u \in C^1(\mathbb{R} \times [0,\infty))$ is a solution of (1), then

$$\int_0^\infty \int_{\mathbb{R}} \left(u(x,t)\varphi_t(x,t) + q(u(x,t))\varphi_x(x,t)\right) dxdt + \int_{\mathbb{R}} g(x)\varphi(x,0)\, dx = 0 \quad \forall \varphi \in C_c^\infty(\mathbb{R}\times[0,\infty)) \tag{2}$$

   (Hint: multiply the first equation in (1) by φ and integrate).

   (b) Conversely, if $u \in C^1(\mathbb{R} \times [0,\infty))$ satisfies (2), prove it is a solution of (1).

   Remark: we say that $u \in L^\infty(\mathbb{R})$ is a *weak solution* of (1) if condition (2) holds true. We just proved that, if u is of class C^1, then the two notions coincide.

   (c) Consider $q(u) = bu$ with $b \in \mathbb{R}$ (transport equation). Assume further that $g \in L^\infty(\mathbb{R})$. Show that $u(x,t) = g(x - bt)$ is a weak solution.

3. Show that the Cauchy problem

$$u_x = 0 \text{ in } \mathbb{R}^2, \qquad u(x, x^3) = x$$

   has no (classical) solution, even though each projected characteristic intersects exactly once the curve $\gamma = \{(x,x^3) : x \in \mathbb{R}\}$. Justify why this is not in contradiction with the existence and uniqueness theorem we have seen in the lecture.

4. Determine the solution to the following Cauchy problems. Plot the projected characteristics and justify on which domains one has uniqueness.

   (a) $\sqrt{1-x^2}u_x + u_y = 0$ with $u(0,y) = y$.

   (b) $yu_x + xu_y = 0$ with $u(0,y) = e^{-y^2}$.

5. Consider the first order linear equation

$$xu_x - yu_y = 0, \qquad \text{for } u = u(x,y).$$

   (a) Show that the projected characteristics are of the form $xy = d$, for $d \in \mathbb{R}$.

   (b) Construct a solution of the equation in the whole $\mathbb{R}^2$ which is *not* of the form $u(x,y) = f(xy)$, for $f \in C^1$. (Hint: it may help to plot the projected characteristics).

   (c) Determine the general solution of the equation.

6. Determine the solution to the following Cauchy problems.

   (a) $u_x + u_y + u = e^{x+2y}$ with $u(x,0) = 0$.

   (b) $u_x + u_y = u^2$, where $u(x,0) = h(x)$, for $h \in C^1(\mathbb{R})$.

(c) $xu_y - yu_x = u$, $u(x,0) = h(x)$, for $h \in C^1(\mathbb{R})$.

(d) $xu_x + u_y + 2zu_z = 3u$, for $u(x,y,0) = g(x,y)$.

7. Prove Theorems 2.15 and 2.24 of the lecture notes.

8. Solve the Cauchy problem

$$\begin{cases} xu_x - yu_y = u - y & \text{for } x, y > 0, \\ u(y^2, y) = y & \text{for } y > 0. \end{cases}$$

Can there exist a solution to the problem in a domain containing the origin?

9. Take $B := \{(x,y) \in \mathbb{R}^2 : \ x^2 + y^2 < 1\}$, and let Ω be an open set containing $\overline{B}$. Let $u \in C_1(\overline{B})$ be a solution of the equation

$$a(x,y)u_x + b(x,y)u_y = \alpha u \quad \text{in } \Omega \quad (\alpha \neq 0),$$

for $a, b \in C^1(\Omega)$ such that

$$a(x,y)x + b(x,y)y > 0 \quad \text{for every } (x,y) \in \partial B.$$

Show that:

(a) the function u cannot admit both its maximum and minimum in B, unless $u \equiv 0$ in $\overline{B}$;

(b) if $\alpha < 0$, then $u \equiv 0$ in $\overline{B}$.

10. (Characteristic variables) Consider the (linear) equation

$$a(x,y)u_x + b(x,y)u_y + c(x,y)u = f(x,y) \tag{3}$$

where a, b, c, f are C^1 functions in an open set $\Omega \subset \mathbb{R}^2$, with $a^2 + b^2 \neq 0$ in Ω. Let φ be a first integral of the system

$$\frac{dx}{dt}(t) = a(x(t), y(t)), \quad \frac{dy}{dt}(t) = b(x(t), y(t))$$

in a neighborhood of a point $(x_0, y_0) \in \Omega$. Assume there exists $\psi(x,y)$ such that

$$\begin{vmatrix} \varphi_x & \varphi_y \\ \psi_x & \psi_y \end{vmatrix} \neq 0 \qquad \text{in } \Omega$$

(that is, φ and ψ are functionally independent).

(a) Show that the change of variables

$$\xi = \varphi(x,y), \quad \eta = \psi(x,y)$$

turns (3) into an equation of the form

$$w_\eta + P(\xi,\eta)w = Q(\xi,\eta),$$

(b) Deduce a formula for the general solution of (3).

11. Consider a (infinite) thin tube, placed along the $x-$axis, containing a fluid moving in the positive direction. Denote by $u = u(x,t)$ the fluid density at the point x and time t, and suppose that:

- the speed at each point depends on the density in the following way: $v(u) = u/2$;
- the tube's wall is made of a porous material that leaks material to the outside with a rate of κu^2, for some constant $\kappa > 0$.

Derive the PDE that models this problem and solve it with initial condition $u(x,0) = 1$, plotting the projected characteristics. Interprete the result.

12. Construct a weak solution to the Burgers equation $uu_x + u_t = 0$ with initial data

$$h(x) = \begin{cases} 1 & x \leqslant 0 \\ 1 - x & 0 < x < 1 \\ 0 & x \geqslant 1. \end{cases}$$

13. Consider the problem

$$uu_x + u_t = 0, \quad x \in \mathbb{R}, t > 0; \quad u(x,0) = \begin{cases} -1 & x < 0 \\ 1 & x > 0 \end{cases}.$$

Show that, for any $\alpha > 1$, the function

$$u_\alpha(x,t) = \begin{cases} -1 & 2x < -(\alpha+1)t \\ -\alpha & -(\alpha+1)t < 2x < 0 \\ \alpha & 0 < 2x < (\alpha+1)t \\ 1 & (\alpha+1)t < 2x \end{cases}$$

is a weak solution. [Observe that, in this case, we have infinitely many solutions!]