Partial Differential Equations
IST 2022/2023

Problem sheet 1 - Warm up

In what follows, Ω is an open subset of $\mathbb{R}^N$.

1. (Fundamental Lemma of Calculus of Variations) Let $f \in C(\Omega)$ be such that
$$\int_\Omega f\phi = 0 \quad \forall \phi \in \mathcal{D}(\Omega) := C_c^\infty(\Omega).$$
Then $f \equiv 0$ in Ω.

2. (Integration by parts) Take $u, v \in C^2(\overline{\Omega})$, where Ω is an open bounded regular set in $\mathbb{R}^N$. Let $\nu(x) = (\nu_1(x), \ldots, \nu_N(x))$ be the unit outer normal at the point $x \in \partial\Omega$. Derive, from the Divergence Theorem, the following integration by parts formula

   (a) $\displaystyle\int_\Omega u_{x_i} v \, dx = -\int_\Omega u v_{x_i} \, dx + \int_{\partial\Omega} uv\nu_i d\sigma$

   as well as the Green identities:

   (b) $\displaystyle\int_\Omega \Delta u \, dx = \int_{\partial\Omega} \frac{\partial u}{\partial \nu} \, d\sigma,$

   (c) $\displaystyle\int_\Omega \Delta u v \, dx = -\int_\Omega \nabla u \cdot \nabla v \, dx + \int_{\partial\Omega} \frac{\partial u}{\partial \nu} v \, d\sigma,$

   (d) $\displaystyle\int_\Omega \Delta u v - u\Delta v \, dx = \int_{\partial\Omega} \frac{\partial u}{\partial \nu} v - u\frac{\partial v}{\partial \nu} \, d\sigma.$

3. (Hölder's inequality) The purpose of this exercise is to prove Hölder's inequality: for $p \in [1, \infty]$, we have
$$f \in L^p(\Omega),\ g \in L^{p'}(\Omega) \implies fg \in L^1(\Omega) \ \text{ and } \ \|fg\|_1 \leqslant \|f\|_p \|g\|_{p'}.$$

   (a) Prove the case $p = 1$ directly: $fg \in L^1(\Omega)$ and $\|fg\|_1 \leqslant \|f\|_1 \|g\|_\infty$.

   For the case $p > 1$, proceed in the following manner:

   (b) Show Young's inequality: $ab \leqslant \frac{a^p}{p} + \frac{b^{p'}}{p'}$ for all $a, b > 0$.
   [Suggestion: apply the logarithmic function to both sides and use its concavity]

   (c) By applying Young's inequality with $a = |f(x)|$ and $b = |g(x)|$, deduce that $fg \in L^1(\Omega)$ and $\|fg\|_1 \leqslant \dfrac{1}{p}\|f\|_p^p + \dfrac{1}{p'}\|g\|_{p'}^{p'}$.

   (d) Obtain Hölder's inequality by applying the previous item to λf and g and by choosing λ appropriately.

4. (Young's theorem) Young's theorem states that
$$f \in L^1(\mathbb{R}^N),\ g \in L^p(\mathbb{R}^N) \Rightarrow f * g \in L^p(\mathbb{R}^N) \text{ and } \|f * g\|_p \leqslant \|f\|_1 \|g\|_p.$$
Prove the last part of the theorem, *i.e.*, the inequality $\|f * g\|_p \leqslant \|f\|_1 \|g\|_p$, for:

   (a) $p = \infty$

   (b) $p = 1$

   (c) $p = 2$, using the decomposition $|f(x-y)| = |f(x-y)|^{\frac{1}{2}} |f(x-y)|^{\frac{1}{2}}$ and Cauchy-Schwarz's inequality.

5. Use Lebesgue's dominated convergence theorem to show that

$$f \in C_c(\mathbb{R}^N),\ g \in L^1_{loc}(\mathbb{R}^N) \Rightarrow f * g \in C(\mathbb{R}^N).$$

Is it true that $f * g \in C_c(\mathbb{R}^N)$ for any f and g as above?

6. Take $f \in L^1(\mathbb{R}^N)$ and $g \in L^p(\mathbb{R}^N)$, with $1 \leqslant p \leqslant \infty$. Show that

$$\mathrm{spt}\,(f * g) \subseteq \overline{\mathrm{spt}\,(f) + \mathrm{spt}\,(g)}.$$

[Remark: In case you do not know yet what is an L^p-space, prove the result for continuous functions only, in order to start practicing the notion of convolution.]