Partial Differential Equations

2025/2026

2^{nd} Exam - 2 July 2026 - 13:00

Duration: 2 hours

(3/20) **1.** Using the method of characteristics, solve the Cauchy problem

$$\begin{cases} u_t + x\,u_x = u^2, & x \in \mathbb{R},\ t > 0, \\ u(x,0) = \dfrac{1}{1+x^2}, & x \in \mathbb{R}, \end{cases}$$

indicating the largest time interval where the solution is defined.

2. Let $\Omega \subset \mathbb{R}^N$ be a connected bounded open set, and suppose that $u \in C^2(\Omega) \cap C(\overline{\Omega})$ is **superharmonic** in Ω, that is, suppose that $\Delta u \leq 0$ in Ω.

(2/20) (a) Show that

$$u(x) \geq \frac{1}{|\partial B_r|} \int_{\partial B_r(x)} u(y) d\sigma(y)$$

for all $x \in \Omega$ and all $r > 0$ such that $B_r(x) \subset \Omega$.

(1/20) (b) Conclude that $\min_{x\in\overline{\Omega}} u(x) = \min_{x\in\partial\Omega} u(x)$.

(3/20) **3.** Let $\Omega \subset \mathbb{R}^N$ be a connected bounded open set with C^1 boundary, and let $f \in C^2(\Omega)$. Consider the initial boundary value problem

$$\begin{cases} u_t - \Delta u = 0, & x \in \Omega,\ t > 0, \\ u(x,0) = f(x), & x \in \Omega, \\ u(x,t) = 0, & x \in \partial\Omega,\ t > 0. \end{cases}$$

Use the L^2 energy

$$E(t) = \frac{1}{2} \int_\Omega u^2(x,t)\, dx$$

to prove that there exists at most one classical solution

$$u \in C^{2,1}(\overline{\Omega} \times [0,\infty)).$$

4. Consider the tempered distribution $T \in \mathcal{S}'(\mathbb{R})$ given by

$$\langle T, \varphi \rangle_{\mathcal{S}' \times \mathcal{S}} = \int_{\mathbb{R}} \frac{\varphi(x) - \varphi(0)}{x} \, dx.$$

(1/20) (a) Show that $xT = 1$.

(2/20) (b) Show that

$$\widehat{T} = -2i\pi H(\xi) + C$$

for some constant $C \in \mathbb{C}$, where H denotes the Heaviside function. You may use without proof the fact that distributions whose derivative vanishes are represented by constant functions.

5. Decide whether each statement is true or false, and explain why.

(1/20) (a) If $f \in H^1(\mathbb{R}^3)$ then $f \in L^6(\mathbb{R}^3)$.

(1/20) (b) The function $f(x) = \dfrac{|x|^{\frac{1}{2}}}{1 + x^2}$ is in $W^{1,1}(\mathbb{R})$.

(1/20) (c) If $f \in L^1(\mathbb{R})$ and $g(x) = \displaystyle\int_{-\infty}^{x} f(t) \, dt$ is also in $L^1(\mathbb{R})$ then $g \in W^{1,1}(\mathbb{R})$.

6. Consider the boundary value problem

$$\begin{cases} -2\dfrac{\partial^2 u}{\partial x^2} - 2\dfrac{\partial^2 u}{\partial x \partial y} - 3\dfrac{\partial^2 u}{\partial y^2} + u = f & \text{in } \Omega, \\ (2u_x + u_y)x + (u_x + 3u_y)y = g & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^2$ is the open unit ball centered at the origin, $f \in L^2(\Omega)$, and $g \in L^2(\partial\Omega)$.

(2/20) (a) Show that the weak formulation of the problem is: find $u \in H^1(\Omega)$ such that

$$B(u, v) = \phi(v) \qquad \forall\, v \in H^1(\Omega),$$

where

$$B(u, v) = \int_{\Omega} (2u_x v_x + u_x v_y + u_y v_x + 3u_y v_y + uv) \, dx \, dy$$

and

$$\phi(v) = \int_{\Omega} fv + \int_{\partial\Omega} gv.$$

(1/20) (b) Use the trace theorem to show that ϕ is a continuous linear functional on $H^1(\Omega)$.

(2/20) (c) Use the Lax–Milgram theorem to prove that the problem admits a unique weak solution.