Partial Differential Equations

2025/2026

1st Exam - 18 June 2026 - 13:00
Duration: 2 hours

(3/20) **1.** By using the method of characteristics, find the unique function $u \in C^1(\mathbb{R}^2)$ satisfying

$$\begin{cases} u_x + y\, u_y = u \\ u(0,y) = y^2 \end{cases} .$$

(3/20) **2.** Let $\Omega \subset \mathbb{R}^N$ be an open set, and let $\{u_k\}_{k\in\mathbb{N}}$ be a sequence of harmonic functions on Ω such that $\|u_k - u\|_{L^\infty(\Omega)} \to 0$ for some continuous function $u \in C(\Omega)$. Prove that u is harmonic on Ω.

(3/20) **3.** Consider the following initial value boundary problem the **telegraph equation:**

$$\begin{cases} u_{tt} + \gamma\, u_t - \Delta u = 0, & x \in \Omega, \quad t > 0, \\ u(x,0) = f(x), & x \in \Omega, \\ u_t(x,0) = g(x), & x \in \Omega, \\ u(x,t) = 0, & x \in \partial\Omega, \quad t > 0, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with C^1 boundary, $f \in C^2(\Omega)$, $g \in C^1(\Omega)$ and $\gamma > 0$. Prove that there is at most one solution $u \in C^2(\overline{\Omega} \times [0,+\infty))$ to this problem.

(3/20) **4.** Let $T \in \mathcal{D}'(\mathbb{R})$ be a distribution such that $T' = 0$ in the sense of distributions. Prove that T is represented by a constant function, that is, prove that there exists a constant $c \in \mathbb{R}$ such that

$$\langle T, \varphi \rangle = c \int_{\mathbb{R}} \varphi(x)dx \qquad \forall \varphi \in \mathcal{D}(\mathbb{R}).$$

Hint: Start by showing that $\psi \in \mathcal{D}(\mathbb{R})$ is of the form $\psi = \varphi'$ for some $\varphi \in \mathcal{D}(\mathbb{R})$ if and only if

$$\int_{\mathbb{R}} \psi(x)dx = 0.$$

5. Decide whether each statement is true or false, and explain why.

(1/20) (a) The function $f(x) = e^{-|x|}$ is in $H^2(\mathbb{R})$.

(1/20) (b) If $f \in L^1(\mathbb{R}^N)$ then $\hat{f} \in C(\mathbb{R}^N)$.

(1/20) (c) If $f \in H^1(\mathbb{R}^3) \cap C^\infty(\mathbb{R}^3)$ then $\lim_{\|x\| \to +\infty} f(x) = 0$.

6. Consider the boundary value problem in $\mathbb{R}^N$ given by

$$-\Delta u + \frac{x}{1+|x|^2} \cdot \nabla u + u = \frac{1}{1+|x|^{2N}}$$

together with the condition that u vanishes at infinity.

(2/20) (a) Show that the weak formulation of the problem is: find $u \in H^1(\mathbb{R}^N)$ such that

$$B(u,v) = \phi(v) \qquad \forall v \in H^1(\mathbb{R}^N),$$

where

$$B(u,v) = \int_{\mathbb{R}^N} \nabla u \cdot \nabla v \, dx + \int_{\mathbb{R}^N} \frac{(x \cdot \nabla u)\, v}{1+|x|^2} \, dx + \int_{\mathbb{R}^N} u\, v \, dx$$

and the linear functional $\phi \in H^{-1}(\mathbb{R}^N)$ is defined by

$$\phi(v) = \int_{\mathbb{R}^N} \frac{v}{1+|x|^{2N}} \, dx.$$

(3/20) (b) Use the Lax-Milgram theorem to prove that the problem admits a unique weak solution.

(1/20) (c) What can you say about the regularity of the weak solution?