

Lecture 5

12/11/2015

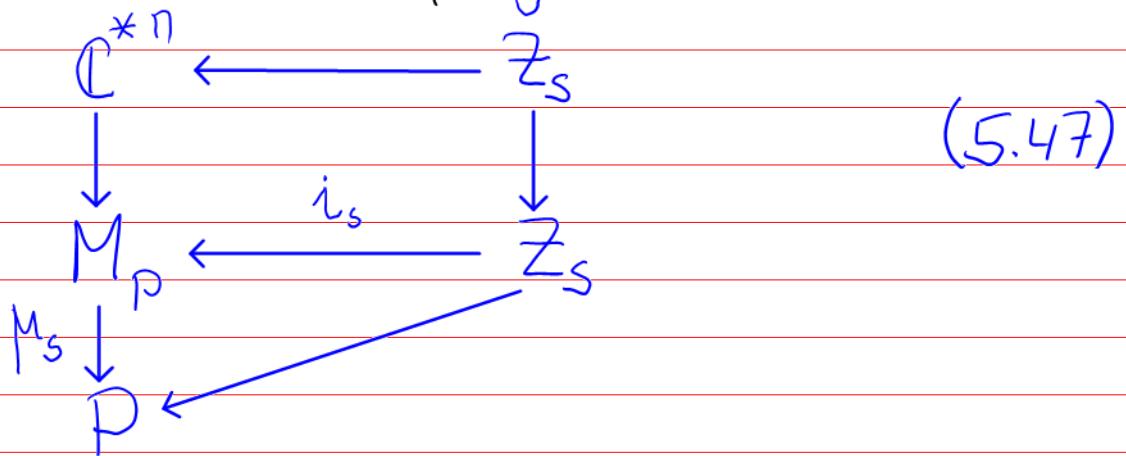
S5

(continuation)

5.4

Choice of $k_0 = g_0 \in C^0(M_p)$, the geodesic k_s and tropicalization [FMNIS]

Consider again the general problem of section 5.1 with an algebraic hypersurface \tilde{Z} in C^{*n} with Newton polytope P assumed to be Delzant. Then we get a compactification as in (5.2) and we are interested in finding a geodesic k_s in the space of Kähler metrics that leads to Kähler tropicalization of (5.2) analogous to what we obtained in the previous section



of the toric manifold

$$\lim_{s \rightarrow \infty} \left(M_p, \frac{\tilde{Z}_s}{s} \right) \quad (5.48a)$$

(Gromov-Hausdorff limit)

and of the compact amoebas

$$\lim_{s \rightarrow \infty} \mathcal{A}_s = \lim_{s \rightarrow \infty} M_s(Z_s) \quad (5.48b)$$

(Hausdorff limit)

The case of K_0 a strictly convex smooth function of the x 's (= components of the moment map) was fully studied in the section 5.3 [B11]. While the tropicalization of the toric manifold was satisfactory (see (5.37)), the tropicalization of the compact amoebas occurs only, in general, in the interior P of the polytope (at the boundary of P the tropical amoeba may get smashed) according to (5.44) and (5.45) (see figures (5.46)).

The reason for this was evident in (5.42) and (5.43) where one sees that the Legendre transform L_H of the moment map ($L_H = \text{id}_P$ if $H = \frac{1}{2}x_1^2 + \dots + x_n^2$) only approximates the tropicalizing \log/s -map for $s \gg 0$ and points $x \in P$. For points $x \in \partial P$ the logarithmic divergence of ∂g_0 leads to the observed smashings.

Let us now show that by choosing a specific continuous non smooth at the compactification divisor $M_p \setminus \tilde{M}_p$ K_0 , namely $K_0 = g_0$ where g_0 is the symplectic potential for the initial Kähler potential K_0 i.e.

$$\dot{K}_0(x) = g_0(x) = \left(\sum_{j=1}^n x_j y_j - K_0(e^y) \right) \Big|_{y = \frac{\partial g_0}{\partial x}} \quad (5.49)$$

We get a "geodesic" of cone angle metrics which however has the very important property of defining a continuous moment map globally defined on or M_p (and C^∞ on the relative interiors of all $\mu_s^{-1}(F)$ where F is any

positive dimensional face of P) and such that on each $\mu_s^{-1}(F)$ coincides with the corresponding Log_s 's map up to a s -independent Legendre transform.

So we can say that μ_s compactifies and joins in a single one all the Log_s 's maps of all faces and thus describes a continuous transition from Z_s to the collection of its tropical amoebas

$$Z_s \leftrightarrow \left\{ A_F^{\text{trop}}, F \in \mathcal{F}_P \right\} \quad (5.50a)$$

in the sense that

$$\begin{aligned} \lim_{s \rightarrow \infty} \mu_s(Z_s \cap \mu_0^{-1}(F)) &= L_U^F \left(\lim_{s \rightarrow \infty} \frac{\text{Log}^F}{s} (Z_s \cap \mu_0^{-1}(F)) \right) \\ &= L_U^F(A_F^{\text{trop}}) \end{aligned} \quad (5.50b)$$

So we choose an initial Kähler potential $K_0 \in C^\infty(\mathbb{C}^{*n})$ (see (5.7), (5.10)) and let g_0 be the corresponding symplectic potential (see (5.17), (5.22))

$$g_0(x) = \frac{1}{2} \sum_{j=1}^d \ell_j(x) \log(\ell_j(x)) + f(x) \quad (5.51)$$

Then we choose

$$\dot{K}_0(x) = g_0(x) \in \mathcal{C}^0(P) \quad (5.52)$$

and substitute in the geodesic equations (5.27). Then theorem 5.2 gives

us (at least on \tilde{M}_p) the solution of the eqs

$$\begin{aligned} g_s(x) &= g_0(x) + s g_0(x) = \\ &= (1+s) g_0(x). \end{aligned} \quad (5.53)$$

For K_s we obtain an explicit expression

$$K_s(y) = L(g_s) = L((1+s)g_0) \Leftrightarrow$$

$$K_s(y) = \tilde{s} K_0\left(\frac{y}{\tilde{s}}\right) \quad (5.54)$$

$$s = S+1$$

For an initial Bergman potential (5.7) the geodesic reads

$$K_s(y) = \frac{\tilde{s}}{2} \log \left(\sum_{m \in \mathbb{Z}^n} |c_m|^2 e^{2 \frac{m \cdot y}{\tilde{s}}} \right) \quad (5.55)$$

Rmk The metrics γ_s corresponding to (5.55) have cone angle singularities at the toric divisors with cone angles $2\pi/\tilde{s}$ in the directions transverse to the divisors (see [Do12]). □

Tropicalization

To show the Kähler tropicalization of M_p we can use (5.23) to obtain from (5.53)

$$\begin{aligned} \frac{1}{\tilde{s}} \gamma_s|_{M_p} &= \text{Hess } g_0 dx^2 + \frac{1}{\tilde{s}^2} (\text{Hess } g_0)^{-1} d\theta^2 \\ &\xrightarrow[G \rightarrow \infty]{s \rightarrow \infty} \text{Hess } g_0 dx^2 \end{aligned} \quad (5.56)$$



$\text{C}^{\circ}\text{ trop.}$

$\text{C}^{\circ}\text{ trop}$

(5.57)

For the C° -kähler tropicalization of compact amoebas we need the properties of the moment map

$$\begin{aligned} \mu_s &= \frac{\partial}{\partial y} \tilde{s} K_0(y/\tilde{s}) = \\ &= \left. \frac{\partial}{\partial \tilde{y}} K_0(\tilde{y}) \right|_{\tilde{y} = \frac{y}{s}} \end{aligned} \quad (5.58)$$

Thm 5.4 (Florentino-M-Nunes '15)

1. The difference between the Log/\tilde{s} map and the μ_s map (5.58), on M_p , is the s -independent map L_{u_0} , i.e.

$$\begin{array}{ccc} M_p \cong (\mathbb{C}^*)^n & & (5.59) \\ \downarrow \text{Log} & & \\ P \leftarrow L_{u_0} : \mathbb{R}^n & & \end{array}$$

$$\mu_s = L_{u_0} \circ \frac{\text{Log}}{\tilde{s}}$$

$$L_{u_0} : y \mapsto \frac{\partial u_0}{\partial y}$$

$\tilde{s} = s+1$ and, as before, $u_0(y) = K_0(e^y)$

2. μ_s extends to a continuous map

$$\mu_s : M_p \rightarrow P \quad (5.60)$$

which is C^∞ in the relative interior of $\mu_0^{-1}(F)$ for all faces of P .

3. The diagram (5.59) is valid for the restriction of μ_s to $\mu_0^{-1}(F)$ and for the $\text{Log}^F - \tilde{s}$ map

$$\text{Log}^F : \mu_0^{-1}(F) \rightarrow \mathbb{R}^{n_F} \quad (5.61)$$

defined for any choice of point $p_F \in \mu_0^{-1}(F)$ and of generators of the n_F -dimensional subtorus acting freely on $\mu_0^{-1}(F)$ and for all positive dimensional faces F

$$\begin{array}{ccc} & \mu_0^{-1}(F) \cong (\mathbb{C}^*)^{n_F} & \\ \mu_s^F \swarrow & \downarrow \text{Log}^F & \\ P \leftarrow L_{U_0^F} & \mathbb{R}^{n_F} & \end{array} \quad (5.62)$$

$$\mu_s^F = L_{U_0^F} \circ \frac{\text{Log}^F}{\tilde{s}}, \forall F$$

where $\mu_s^F = \mu_s|_{\mu_0^{-1}(F)}$ and U_0^F is the pullback to $\mu_0^{-1}(F)$ of the Kähler potential defined in vertex coordinate charts of any vertex $m_0 \in F$.



Theorem 5.4 shows that indeed the moment map μ_s corresponding to the C^∞ geodesic (5.54) compactifies the Log^F / \tilde{s} maps of all faces of the polytope P to a single con-

tinuous map. We have then (5.50) and the restriction of the limit of the compact amoeba to the relative interior of every face coincides with the Legendre transform of the tropical amoeba

$$\mathcal{A}_\infty = \lim_{s \rightarrow \infty} \mathcal{A}_s = \lim_{s \rightarrow \infty} \mu_s(z_s) \quad (5.63)$$

$$\mathcal{A}_\infty \cap F = L_{UF}(\mathcal{A}_F^{\text{trop}})$$

5.5

Statistical physics interpretation:
C° Kahler / Tropicalization and the
 $K_{\text{Bol}} \rightarrow \infty$ limit

To be added

References for Lectures 4-5

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