

Lecture 4

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§5

Kähler Tropicalization of Toric varieties and of hypersurfaces in toric varieties

5.1

Summary

Consider an algebraic hypersurface

$$\check{Z} = \left\{ w \in \mathbb{C}^{*n} : \sum_{m \in P \cap \mathbb{Z}^n} a_m w^m = 0 \right\} \subset (\mathbb{C}^{*})^n \quad (5.1)$$

We will assume that P is a Delzant polytope and let M_P denote the algebraic compactification of \mathbb{C}^{*n} associated with P . We obtain

$$\begin{array}{ccc} \mathbb{C}^{*n} & \xhookrightarrow{\quad} & \check{Z} \\ \downarrow & & \downarrow \\ M_P & \xhookrightarrow{\quad} & Z \end{array} \quad (5.2)$$

As in the examples of §3, in order to get a Kähler tropicalization of (5.2) we choose an initial Kähler potential on \mathbb{C}^{*n}

$$k_0 \in C^\infty(\mathbb{C}^{*n}) \quad (5.3)$$

and an initial velocity, globally defined on M_P

$$\text{In 5.3: } \dot{k}_0 \in C^\infty(M_P)$$

$$\text{In 5.4: } \dot{k}_0 = g_0 \in C^0(M_P),$$

where g_0 denotes the symplectic potential corresponding to k_0 ,

$$g_0 = \omega_{\mu_0}, \\ \mu_0(y) = k_0(e^y)$$

If k_0 and μ_0 are both toric we will see that the solution k_s of the geodesic equations with these initial data is given by the Legendre transform of a straight line in the space of symplectic potentials

$$k_s = \omega(g_0 + s k_0) \quad (5.4)$$

Then, by Kähler tropicalization of (M_P, γ_s) , where γ_s denotes the metric corresponding to k_s , we will mean that, if $k_0 = \tilde{H} \circ \mu_0$, where μ_0 is the moment map corresponding to k_0 , and \tilde{H} is strictly convex on P , then

$$\lim_{s \rightarrow \infty} (M_P, \frac{\gamma_s}{s}) \underset{\substack{\text{GH} \\ \text{up}}} \longrightarrow (\overset{\circ}{P}, \text{Hess}_x(\tilde{H}) dx^2). \quad (5.5)$$

By Kähler tropicalization of Z^{CM}_P we will mean that the Hausdorff limit of the compact amoebas will exhibit tropical behaviour

$$A_s := \mu_s(Z_s)$$

$$\lim_{s \rightarrow \infty} A_s = A_\infty \quad \begin{matrix} \text{study relation} \\ \text{with} \end{matrix} \quad (5.6)$$

$$A^{\text{trop}} = \lim_{s \rightarrow \infty} \frac{\log(Z_s)}{s}$$

where we have assumed some dependence of the coefficients of the polynomial defining \mathbb{Z}^n on s (in order to get interesting tropical amoebas).

5.2 Choice of k_0 .

As we will see the choice of k_0 will not affect neither the tropicalization of M_P nor the tropicalization of $\mathbb{Z}^n \subset M_P$. They both will depend only on k_0 .

In any case for k_0 we can take a toric Bergman potential i.e. a potential obtained by taking the pull-back of the Fubini-Study Kähler potential wrt a toric embedding of M_P into $\mathbb{C}\mathbb{P}^N$, $N = |\mathbb{P}^n \mathbb{Z}^n| - 1$,

$$k_0^B(w) = (F^* k_{FS})(w) = \\ = \frac{1}{2} \log \left(\sum_{m \in \mathbb{Z}^n \cap P} |c_m|^2 |w^m|^2 \right), \quad (5.7)$$

where we are assuming that the line bundle associated with P is very ample (if it is only ample we can take ℓ_P instead of P and substitute $1/2$ by $1/2e$ in front of the logarithm in the rhs of (5.7)). Also the coefficients of all the vertices of P have to be different from zero (see Gu 94)

$$c_{m_0} \neq 0 \quad \forall m \in \text{Vert}(P) \quad (5.8)$$

The most general form of a toric kähler potential K such that

$$[\bar{i} \partial \bar{\partial} K] = [\bar{i} \partial \bar{\partial} K_B] \quad (5.9)$$

is then given by

$$K_0 = K_0^B + f \quad (5.10)$$

where f is a global T^n -invariant cs function on M_p such that K_0 is strictly plurisubharmonic (spsh) i.e.

$$i \partial \bar{\partial} K_0 > 0 \quad (5.11)$$

Now, since

$$K_0(w) = K_0(|w|) = K_0(e^y) =: u_0(y),$$

we obtain

$$\omega_0 = 2i \partial \bar{\partial} K_0 = \sum_{ij} \underbrace{\frac{\partial^2 u_0(y)}{\partial y_i \partial y_j}}_{>0} dz_i \wedge d\bar{z}_j \Leftrightarrow \quad (5.12)$$

$$\omega_0 = \sum_{ij} \frac{\partial^2 u_0}{\partial y_i \partial y_j} dy_i \wedge d\theta_j = \sum_{j=1}^r dx_j \wedge d\theta_j \quad (5.13)$$

$$= \sum_{ij} \frac{\partial x_j}{\partial y_i} dy_i \wedge d\theta_j = \sum_{j=1}^r dx_j \wedge d\theta_j$$

So that the functions $x_i = \frac{\partial u_0}{\partial y_i}$ generate the torus action and therefore are components of the

moment map,

$$X_{x_j} = - \frac{\partial}{\partial \theta_j} \Leftrightarrow \mu_0(w) = (x_1, \dots, x_n) = \left(\frac{\partial u_0}{\partial y_1}, \dots, \frac{\partial u_0}{\partial y_n} \right) \quad (5.14)$$

As we saw in (5.12) the Kähler potential u_0 is strictly convex and therefore the Legendre transform it generates

$$L_{u_0}: y \mapsto x = \frac{\partial u_0}{\partial y} \quad (5.15)$$

is a diffeomorphism to its image \check{P} and its inverse is also a Legendre transform associated with the dual potential g_0 .

$$\mathbb{R}^n \xleftrightarrow[L_{g_0}]{L_{u_0}} \check{P}, \quad (5.16)$$

where g_0 is (5.17)

$$g_0 = L(u_0) = \left(\sum x_j y_j - u_0(y) \right) \Big|_{y=y(x)}$$

Before we state Guillemin-Abreu theorem on symplectic potentials let us compute, from (5.7) and (5.14), the moment map for a Bergman Kähler potential

$$x = \mu_0 = \frac{\partial u^B}{\partial y} = \sum_{m \in P \cap \mathbb{Z}^n} m p_m(y) \in \overset{\cup}{P} \quad (5.18)$$

where the "Gibbs distribution" factors define a y -dependent probability distribution on the integral points of P (see [Ka11])

$$1 < p_m(y) = \frac{|c_m| e^{z_m \cdot y}}{\sum |c_e|^2 e^{2e \cdot y}} \quad (5.19)$$

$$\sum_{m \in P \cap \mathbb{Z}^n} p_m(y) = 1$$

$$x = \mu_0(w) = \mathbb{E}_y(\mu)$$

Summarizing we obtained the following diagram

$$\begin{array}{ccc}
 & C^{*n} & \\
 & \downarrow \text{Log} & \\
 \mu_0 & \nearrow & \\
 (\text{Lie } T^n)^* \cong \mathbb{R}^n \overset{\cup}{\supset} P & \xleftarrow{\text{L}_{\mu_0}} & \mathbb{R}^n \cong \text{Lie } T^n \\
 & \downarrow \text{L}_{g_0} &
 \end{array} \quad (5.20)$$

Thm 5.1

Guillemin '94, Abreu '98

1. Let

$$P = \{x \in \mathbb{R}^d : \ell_i(x) = \langle v_i, x \rangle + \lambda_i \geq 0, i=1, \dots, d\} \quad (5.21)$$

then

$$g_0(x) = g_P(x) + f(x), \quad (5.22)$$

$$g_P(x) = \frac{1}{2} \sum_{j=1}^d \ell_j(x) \log(\ell_j(x)),$$

$f \in C^\infty(P)$ and $\text{Hess}_x(g_p + f) \geq 0$

(2.) $\gamma_0 = G_0 dx^3 + G_0^{-1} d\theta^2 =$ (5.23)
 $= \sum_{j,k} (G_0)_{jk} dx_j dx_k + (G_0^{-1})_{jk} d\theta_j d\theta_k$

where $G_0 = \text{Hess}_x g_0$

(3.) $L_{g_0}: \tilde{P} \rightarrow \mathbb{R}^n$ is the inverse
map (diffeomorphism) of L_{u_0}

$$L_{u_0}: y \mapsto x = \frac{\partial u_0}{\partial y} \quad (5.24a)$$

$$L_{g_0}: x \mapsto y = \frac{\partial g_0}{\partial x} \quad (5.24b)$$

Therefore the complex structure on
the symplectic coordinates is defined
by the holomorphic coordinates

$$w = e^{y+i\theta} = e^{\frac{\partial g_0}{\partial x}} + i\theta \quad (5.25)$$



5.3 Choice of $k_0 \in C^\infty(M_P)$, the geodesic k_s and tropicalization [FMN15]

We choose for k_0 a global T^n -invariant function on M_P

$$\dot{k}_0 = H \in C^\infty_{T^n}(M_P) \quad (5.26)$$

and then have to find the unique geodesic k_s with initial conditions

K_0, K_s (5.10), (5.26). The toric case is the few known solutions of the geodesic equations on the space of Kähler metrics.

Thm 5.2 [Lempert '85 ?]

The solution of the Cauchy problem for the geodesics on the space of toric Kähler metrics on M_P

$$\begin{cases} \ddot{K}_s = -\frac{1}{2} \|\nabla^{K_s} \dot{K}_s\|_{K_s}^2 \\ K_0 \\ \dot{K}_0 = H \end{cases} \quad (5.27)$$

is

$$\begin{aligned} K_s &= L(g_s) = \\ &= \left(\sum_{j=1}^n x_j y_j - g_s \right) \Big|_{y=\frac{\partial g_s}{\partial x}} \end{aligned} \quad (5.28)$$

where g_s is a straight line in the space of symplectic potentials

$$\begin{aligned} g_s(x) &= g_0(x) + s K_0(x) = \\ &= g_0(x) + s H(x) \end{aligned} \quad (5.29)$$

The solution 5.28 is valid for all s :

$$\text{Hess}(g_s) = G_0 + s \text{Hess}_x H \Big|_P > 0 \quad (5.30)$$

Proof

It is very instructive and quite simple to prove this theorem with our 3 step process of finding the imaginary time flow of X_H

$$\varphi_{t_0, \mathcal{J}_0}^{X_H} \quad (5.31)$$

and the corresponding geodesic

$$k_s \text{ (or } \gamma_s) \quad (5.32)$$

as we did in the two previous toric examples: \mathbb{C}^* (3.21), (3.27)
 \mathbb{C} (3.44), (3.47)

Step 1

We just have to find the integrable curves of X_H

$$X_H = - \sum_{j=1}^n \frac{\partial H}{\partial x_j} \frac{\partial}{\partial \theta_j}$$

$$\left. \begin{array}{l} \dot{\theta}_t = - \frac{\partial H}{\partial x} \\ \dot{x}_t = 0 \end{array} \right\} \Rightarrow \varphi_t^{X_H} \quad \left. \begin{array}{l} \theta_t = \theta - t \frac{\partial H}{\partial x} \\ x_t = x \end{array} \right\} \quad (5.33)$$

on local, real analytic functions, we continue analytically in t

$$(\varphi_t^{X_H})^*: \left. \begin{array}{l} \theta \mapsto \theta - t \frac{\partial H}{\partial x} \\ x \mapsto x \end{array} \right\} \quad (5.34)$$

Step 2

We restrict the action (5.34) to just the \mathcal{J}_0 -holomorphic fun-

cations generated by the holomorphic coordinates (5.25) on the dense orbit -

$$\begin{aligned} w &= e^{\frac{\partial g_0}{\partial x}} + i\theta \\ w \mapsto w_{is} &= (\psi_t^{X_H})_{t \mapsto is}^* \left(e^{\frac{\partial g_0}{\partial x}} + i\theta \right) = \\ &= e^{\frac{\partial g_0}{\partial x} + i(\theta - is\frac{\partial H}{\partial x})} \quad \Leftrightarrow \\ (5.34) \end{aligned}$$

$$w_{is} = e^{\frac{\partial g_0}{\partial x} + s\frac{\partial H}{\partial x} + i\theta} = e^{\frac{\partial}{\partial x}(g_0 + sH) + i\theta} \quad (5.35)$$

Comparing (5.35) with (5.25) we already see that w_{is} are the holomorphic coordinates corresponding to the symplectic potential (5.29) which proves the theorem as the other claims are direct consequence of Thm 5.1.

Tropicalization

We are now ready to study the Kähler (i.e. through Kähler geodesics) tropicalization of toric varieties M_p and of hypersurfaces $Z \subset \mathbb{C}P^n$.

First of all from (5.22) we see that g in (5.29) will be defined for all $s \geq 0$ if $\text{Hess } H \geq 0 \forall x \in P$. For tropicalization we will in fact need the inequality to be strict i.e. H to be a strictly convex function of the x 's. From (5.23) we see that

$$\gamma_s = \left(G_0 + s \text{Hess}_x(H) \right) dx^2 + \left(G_0 + s \text{Hess}_x(H) \right)^{-1} d\theta^2 \quad (5.36)$$

i.e.

$$\begin{aligned} \lim_{s \rightarrow \infty} \frac{\gamma_s}{s} &\stackrel{?}{=} \\ &= \lim_{s \rightarrow \infty} \left(\text{Hess}_x(H) + \frac{1}{s} G_0 \right) dx^2 + \frac{1}{s^2} \left(\text{Hess}_x(H) + \frac{1}{s} G_0 \right)^{-1} d\theta^2 \\ &\stackrel{?}{=} \text{Hess}_x(H) dx^2 \end{aligned}$$

Provided that $\boxed{\text{Hess}_x(H) > 0}$, i.e., $K_0 = H$
is a strictly convex function of the x_i .

So that indeed we have Kähler tropicalization of M_p

$$\left(M_p, \frac{\gamma_s}{s} \right) \xrightarrow{s \rightarrow \infty} \left(\overset{\circ}{P}, \text{Hess } H dx^2 \right) \quad (5.37)$$

Let us now study the Kähler tropicalization of hypersurfaces

$$Z_s \subset M_p \quad (5.38)$$

where the dependence of Z_s on s is through a Viro patchworking valuation of coefficients of the polynomial in (5.1) to get more interesting tropical amoebas. For every s the compact amoebas are defined as in [Mioo]

$$A_s = M_s(Z_s) \quad (5.39)$$

The novelty in (5.39) is in the s -dependence obtained by taking a tropicalizing geodesic $K_s(w)$ and the corresponding moment map $M_s(w) = \frac{\partial}{\partial y} K_s(e^y)$ (see (5.4)).

Let us extend the diagram (5.20)

$$\begin{array}{ccccc}
 \mathbb{Z}_s & \hookrightarrow & T^n \times \overset{\circ}{P} & \xrightarrow{\psi_s} & \mathbb{C}^{*n} \supset \mathbb{Z}_s = \{ q(w) = 0 \} \\
 & & \downarrow & \mu_s & \downarrow \text{Log} \\
 & & P & \xleftarrow{L_{g_s}} & \mathbb{R}^n \\
 & & \downarrow & L_{u_s} &
 \end{array} \tag{5.40}$$

where ψ_s is the diffeomorphism

$$\psi_s(\theta, x) = e^{\frac{\partial g_s}{\partial x} + i\Theta} = e^{\frac{\partial g_0}{\partial x} + s \frac{\partial H}{\partial x} + i\Theta} \tag{5.41}$$

We know $g_s = g_0 + sH$ explicitly and we know L_{u_s} and K_s only implicitly. Fortunately to study \mathbb{Z}_s we don't need to know μ_s explicitly as we can use ψ_s in (5.41), pull-back \mathbb{Z}_s to \mathbb{Z}_s and then project on the second component of (θ, x) . \mathbb{Z} is, naturally,

$$\mathbb{Z}_s = \left\{ (\theta, x) \in T^n \times \overset{\circ}{P} : \sum_{m \in P \cap \mathbb{Z}^n} a_m(s) e^{m \cdot \left(\frac{\partial g_0}{\partial x} + s \frac{\partial H}{\partial x} + i\Theta \right)} = 0 \right\} \tag{5.42}$$

We see that in $\overset{\circ}{P}$

$$\log_s w = \frac{\partial H}{\partial x} + \frac{1}{s} \frac{\partial g_0}{\partial x} \underset{x \in \overset{\circ}{P}}{\underset{s \gg 0}{\approx}} \frac{\partial H}{\partial x} \tag{5.43}$$

And one proves the following thm.

Thm 5.3 (Baier, Florentino, M, Nunes)

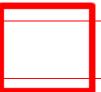
Let $A^{\text{TROP}} = \lim_{s \rightarrow \infty} \frac{\log(z_s)}{s}$. Then

$$L_H(A_\infty) = \pi_{L_H(P)}(A^{\text{TROP}}) \quad (5.44)$$

where L_H is the Legendre transform generated by H . For $H = \frac{1}{2}(x_1^2 + \dots + x_n^2)$ (5.44) simplifies to

$$A_\infty = \pi_P(A^{\text{TROP}}) \quad (5.45)$$

where $\pi_Q : \mathbb{R}^n \rightarrow Q$ is the projection of \mathbb{R}^n onto the convex body Q .



Few examples (see [BII])

