

Lecture 3

9/11/2015

§3 (continuation)

3.3 Kähler tropicalization of \mathbb{C}

$$\omega = e^z = e^{y+i\theta} = \vartheta_1 + i\vartheta_2 \in \mathbb{C} \quad (3.28)$$

The Kähler metric (3.5) we had for \mathbb{C}^*

$$\gamma_0 = dy^2 + d\theta^2$$

does not extend to \mathbb{C} so we choose a also flat T' -invariant metric

$$\gamma_0 = d\vartheta_1^2 + d\vartheta_2^2 \quad (3.29)$$

with T' -invariant Kähler potential

$$K_0(w, \bar{w}) = K_0(|w|) = \frac{1}{2} |w|^2 = \frac{e^{2y}}{2} = u_0(y), \quad (3.30)$$

Note that u_0 is just the T' -invariant Kähler potential seen as a function on $\text{Lie } T'$. Then

$$\begin{aligned} \omega &= z_i \bar{\partial} \partial (K_0) = \underbrace{dw \wedge d\bar{w}}_{-z_i} = \\ &= \frac{2i}{4} \frac{\partial^2}{\partial y^2} (u_0(y)) dz \wedge d\bar{z} \Leftrightarrow \end{aligned}$$

$$\Leftrightarrow \omega = \frac{\partial^2 u_0}{\partial y^2} dy \wedge d\theta \quad (3.31)$$

We see that if we define

$$x = \frac{\partial u_0}{\partial y} = e^{2y} = |w|^2 \quad (3.32)$$

and substitute in (3.31) we obtain

$$\omega = \frac{\partial x}{\partial y} dy \wedge d\theta = dx \wedge d\theta \quad (3.33)$$

and therefore

$$X_x = -\frac{\partial}{\partial \theta} \quad \text{and}$$

$$x = M_0, \quad (3.34)$$

coincides with moment map of the T^1 -action. We therefore obtained that the moment map in ω coordinates is the Legendre transform induced by u_0 (see (3.32)). As we will see this result will continue to hold for any toric manifold.

For K_0 we choose any strictly convex function of $x = M_0$, e.g.

$$K_0(w, \bar{w}) = H = x + \alpha \frac{x^2}{2} = |w|^2 + \alpha \frac{|w|^4}{2} \quad (3.35) \quad \alpha > 0$$

The geodesic K_{γ_s} with initial conditions (3.30), (3.35) will again be found with the three steps leading to the imaginary time flow of X_H

$$\varphi_{is} \equiv \varphi_{is}^{X_H, J_0} \Rightarrow \gamma_s = \omega(\cdot, \varphi_{is}^*(J_0) \cdot) \quad (3.36)$$

Step 1

For X_H we get

$$X_H = -\frac{\partial H}{\partial x} \frac{\partial}{\partial \theta} = -(1+\alpha x) \frac{\partial}{\partial \theta} \quad (3.37)$$

and therefore for its real time flow we obtain the ODE

$$\begin{cases} \dot{\theta}_t = -1 - \alpha x_t \\ \dot{x}_t = 0 \end{cases}$$

↓

$$\psi_t^{X_H} : \begin{cases} \theta_t = \theta - (1+\alpha x)t \\ x_t = x \end{cases} \quad (3.38)$$

So that, for local real analytic functions, we can continue analytically in t

$$(\psi_t^{X_H})^* : \begin{cases} \theta \mapsto \theta - (1+\alpha x) \text{ is } (3.39) \\ x \mapsto x \end{cases}$$

Step 2

We now restrict the action (3.39) on all local real analytic functions to \mathcal{J}_0 holomorphic ones generated by w

$$w = e^{y+i\theta} = \sqrt{x} e^{i\theta} \quad (3.32)$$

$$w \mapsto w_{is} = (\psi_t^{X_H})^* (w) = \sqrt{x} e^{i(\theta - is(1+\alpha x))} \quad (=)$$

$$w_{is} = \sqrt{x} e^{s+s\alpha x} e^{i\theta} \quad (3.40)$$

Now we see that w_{is} defines a complex structure globally on \mathbb{C} if and only if the map

$$\mathbb{R}^+ \rightarrow \mathbb{R} \quad (3.41)$$

$$x \mapsto y_{is} = \frac{1}{2} \log x + s(1+\alpha x)$$

is a diffeomorphism. From

$$\frac{\partial y_{is}}{\partial x} = \frac{1}{2x} + s\alpha \quad (3.42)$$

it is easy to see that (3.41) is a diffeomorphism for $s \geq 0$ and for $s < 0$ maps \mathbb{R}^+ onto $(0, \tilde{R}]$, where

$$\tilde{R} = \frac{1}{\sqrt{2|s|\alpha}} e^{-\frac{1}{2}-\frac{|s|}{\alpha}} \quad (3.43)$$

In fact w_{is} in (3.40) defines a complex structure globally on \mathbb{C} for $s \geq 0$ and on the disc $|w_{is}| < \tilde{R}$ for $s < 0$.

Step 3 The unique diffeo such that
(for $s > 0$)

$$(\psi_{is}^{X_H, J_0})^*: \omega \mapsto w_{is}$$

is

$$w_{is} = \sqrt{c_{is}} e^{i\theta_{is}} = \sqrt{x} e^{s+\alpha x s} e^{i\theta}$$

so that

$$\psi_{is}^{X_H, J_0}(\theta, x) = (\theta, x e^{2s(1+\alpha x)}) \quad (3.44)$$

RMK 3.3

Notice that, for $\alpha = 0$, $H = x$,
 $X_H = -\frac{\partial}{\partial \theta}$, and the action (3.44)

becomes the usual multiplicative action of $\mathbb{R}^+ \mathbb{C}\mathbb{C}^*$ on \mathbb{C} .

$$(\psi_{is}^{X_H, J_0})^*(w) = (\psi_{is}^{X_H})^*(\sqrt{x} e^{i\theta}) = e^s w \quad (3.45)$$



Let us now find the metric γ_s as in the previous example in pages 7 and 8 of lecture 2.

$$\omega = dx \wedge d\theta = \gamma_{1\bar{1}} \frac{dz_{is} \wedge d\bar{z}_{is}}{-2i}, \quad (3.46)$$

where

$$z_{is} = \frac{1}{2} \log(2x) + s(1+\alpha x) + i\theta$$

Then

$$\omega = \gamma_{1\bar{1}}^s \left[\left(\frac{1}{2x} + s\alpha \right) dx + id\theta \right] \wedge \left[\left(\frac{1}{2x} + s\alpha \right) dx + id\theta \right] \Leftrightarrow$$

$$dx \wedge d\theta = \gamma_{1\bar{1}}^s \frac{1+2\alpha s x}{2x} dx \wedge d\theta \Leftrightarrow$$

$$\gamma_{1\bar{1}}^s = \frac{2x}{2x\alpha s + 1}$$

Therefore,

$$\gamma_s = \frac{2x}{2x\alpha s + 1} dz_{is} \cdot d\bar{z}_{is} \Leftrightarrow$$

$$\gamma_s = \frac{2x\alpha s + 1}{2x} dx^2 + \frac{2x}{2x\alpha s + 1} d\theta^2 \quad (3.47)$$

and we see fig 3.2b with the fibers collapsing as $\mathcal{I} \rightarrow \infty$. For $s < 0$ we see that the metric (3.47) is Kähler only for $\chi < 1/2\alpha s$ ($\Leftrightarrow |w_{is}| < R$).

The scalar curvature for γ_s is given by Abreu's formula

$$S_c = -\frac{1}{2} \left(\frac{2x}{2x\alpha s + 1} \right)' = \frac{4\alpha s}{(1+2x\alpha s)^3} \quad (3.48)$$

in agreement with Fig. 3.2b.

To end this example it is instructive to verify that the pair

$$(u_{is}, \gamma_s) = \left((\varphi_{is}^{X_H, \mathcal{I}_0})^{-1}(u), \gamma_s \right) \quad (3.49)$$

satisfies the equation (2.7) in P.6 of Lecture 1, which, in our case, becomes,

$$(\Theta_{is}, \partial_{is}) = -\nabla^{\gamma_s} H \quad (3.50)$$

From (3.35) and (3.47) we obtain

$$\nabla^{\gamma_s} H = \frac{2\alpha}{1+2s\alpha x} (1+\alpha x) \frac{\partial}{\partial x} \quad (3.51)$$

so that (3.50) becomes

$$\begin{cases} \dot{\Theta}_{is} = 0 \\ \dot{x}_{is} = -\frac{2x_{is}(1+\alpha x_{is})}{1+2s\alpha x_{is}} \end{cases} \quad (3.52)$$

This equation has to be satisfied

by the inverse of $\varphi_{is}^{x_i, \theta_0}$ in (3.44), ie.
by $\theta_{is} = \theta$ and x_{is} satisfying

$$x = x_{is} e^{2s(1+\alpha x_{is})} \quad (3.53)$$

Let us use the implicit function theorem to show that x_{is} in (3.53) indeed satisfies the second equation in (3.52).

$$(3.53) \Rightarrow 0 = \frac{\partial x_{is}}{\partial s} + \frac{\partial x_{is}}{\partial s} 2\lambda \alpha x_{is} + 2x_{is}(1+\alpha x_{is})$$

$$\Leftrightarrow \frac{\partial x_{is}}{\partial s} = - \frac{2x_{is}(1+\alpha x_{is})}{1+2s\alpha x_{is}}$$

§4 Actions of G_C on Kähler structures, and their orbits. Symplectic and complex pictures

Let G be the group of Hamiltonian symplectomorphisms of the compact symplectic manifold (M, ω) and G_C its formal complexification. Following Donaldson [Do99] we will think of G_C as a (ω, J) -dependent subset (not subgroup!) of $\text{Diff}(M)$, which we will define below. The Kähler structures are compatible pairs of symplectic forms and complex structures, $(\bar{\omega}, \bar{J})$.

On the space of all Kähler structures there is the standard action of $\text{Diff}(M)$ (not just G_C)

$$P_{st}(\varphi)(\omega, J) = ((\varphi^{-1})^*(\omega), \varphi_*(J)) \quad (4.1)$$

For the studies of stability of Kähler varieties one is however interested in two other "actions" (related with each other), just of G_C . The quotations are meant to remind us that G_C is not a group but we will have the analogue of orbits. In the first action, called symplectic, we act only on the complex structure

$$P_{sy}(\phi)(\omega, J) = (\omega, \phi^* J) \quad (4.2)$$

and in the second, called complex we act only on the symplectic form

$$P_{cx}(\phi)(\omega, J) = ((\phi^{-1})^*\omega, J) \quad (4.3)$$

FOR most (for $\dim_{\mathbb{C}}(M) = n \geq 2$) different morphisms ϕ , the pairs (4.3), of symplectic form and complex structure, are not Kähler so that we restrict these actions to the following subsets of $\text{Diff}(M)$

$$G_C(\omega, J) = \{\varphi \in \text{Diff}(M); p_{\text{ex}}(\varphi)(\omega, J) \text{ is Kähler} \} \quad (4.4)$$

and $[(\varphi^{-1})^*(\omega)] = [\omega]$

These are well defined subsets of $\text{Diff}(M)$ and their orbits in the symplectic picture (when we fix the symplectic form and act on the complex structure) are

$$\begin{aligned} g(\omega, J) &= p_{\text{sy}}(G_C)(\omega, J) = \\ &= \{(\omega, \varphi^*J), \varphi \in G_C\} \end{aligned} \quad (4.5)$$

Assuming that $\text{Aut}(M, J) = \{\text{id}\}$ the action (4.5) is free and therefore

$$g(\omega, J) \cong G_C \quad (4.6)$$

The orbits in the complex picture are

$$\begin{aligned} H(\omega, J) &= p_{\text{ex}}(G_C)(\omega, J) = \\ &= \{((\varphi^{-1})^*\omega, J), \varphi \in G_C\} \end{aligned} \quad (4.7)$$

so that

$$H(\omega, J) \cong G_C / G \quad (4.8)$$

The natural projection

$$\pi: g(\omega, J) \rightarrow H(\omega, J) \quad (4.9)$$

$$(\omega, \varphi^*J) \mapsto ((\varphi^{-1})^*\omega, J)$$

gives as the structure of a principal G -bundle over $\mathcal{H}(\omega, J)$. Notice that Kähler structures on the same fiber of (4.9) are equivalent under the standard action (4.11) of G . Indeed if $\phi \in G$

$$\rho_{sy}(\phi)(\omega, \tilde{\phi}^* J) = (\omega, \phi^* \tilde{\phi}^* J) = \rho_{st}(\phi^{-1})(\omega, \tilde{\phi}^* J)$$

$$\text{and } \pi(\omega, \phi^* \tilde{\phi}^* J) = ((\tilde{\phi} \circ \phi)^{-1})^* \omega, J - (\tilde{\phi}^{-1})^* \omega, J = \pi(\omega, \tilde{\phi}^* J)$$

Rmk 4.1 From the definition of ρ_{sy} and ρ_{cx} it is clear that even though, in each of these actions, either ω or J change by diffeomorphisms the actions ρ_{sy}, ρ_{cx} induced on the metrics do not, in general, correspond to the standard action of a diffeomorphism so that the Kähler structures are not equivalent:

$$\rho_{sy}(\varphi)(\gamma) := \omega(\cdot, \varphi^*(J)\cdot) \neq$$

$$\neq \psi^*(\gamma), \forall \psi \in \text{Diff}(M)$$

for general $\varphi \in G_P \setminus G$ and metric γ , and

$$\rho_{cx}(\varphi)(\gamma) := (\varphi^{-1})^*(\omega)(\cdot, J\cdot) \neq$$

$$\neq \psi^*(\gamma), \forall \psi \in \text{Diff}(M) \quad \square$$

Q1 Why $(G)_c$?

$G_c(\omega, J)$ is defined in (4.11) as a subset of $\text{Diff}(M)$ containing G , but why would this subset have anything to do with the

formal complexification of the group
of Hamiltonian symplectomorphisms of
 (\mathbb{M}, ω) ?

A related question is the following:

Q2- Why are $\alpha(\omega, J)$ and $J(\omega, J)$ of interest in Kähler geometry?

The answer to this question is given by Moser theorem which implies that in fact $J(\omega, J)$ coincides with the set of all Kähler forms in the same cohomology class as ω and compatible with J , which is of course a space of great interest in Kähler geometry (and object e.g. of the Calabi-Yau theorem in its general formulation). From the $\bar{\partial}$ -Lemma we then conclude that the orbit $J(\omega, J)$ is equal to

(4.11)

$$J(\omega, J) = \left\{ \omega_f = \omega + 2i\bar{\partial}^+ f, f \in C^\infty(M) : \omega_f > 0 \right\}$$

The interest of the orbit $\alpha(\omega, J)$ is then a consequence of this space of complex structures being a principal G -bundle over $J(\omega, J)$ as we saw in (4.9).

Now that we have satisfactorily answered to Q2 we can go back to Q1.

From (4.11) we see that the tangent space to $J(\omega, J)$ at every ω_f is naturally isomorphic to $\sqrt{-1}\text{Lie}(G) = \sqrt{-1}C_0^\infty(M, \omega)$, where

$$C_0^\infty(M, \omega) = \left\{ h \in C^\infty(M) : \int_M h \omega^n = 0 \right\} \quad (4.11)$$

$$\begin{aligned} \mathbb{F}_1 \text{Lie } G_r &\longrightarrow T_{\omega_f^r} \mathcal{H} \\ \mathbb{F}_1 H &\longmapsto z \sqrt{-1} \bar{\partial} H, \end{aligned} \tag{4.12}$$

as expected from (4.8). (4.8) however should imply that the vector space isomorphism (4.12) should be related with the expected infinitesimal "action" of G_r on $\mathcal{H}((\omega, J))$.

$$iH \mapsto iX_H^{\omega_f} \mapsto JX_H^{\omega_f} = \nabla^{\omega_f} H \mapsto L_{JX_H^{\omega_f}}(\omega_f) \tag{4.13}$$

Indeed,

$$\boxed{\text{Prop. 4.1}} \quad L_{JX_H^{\omega_f}}(\omega_f) = z \bar{i} \bar{\partial} H. \tag{4.14}$$

Proof Notice that

$$z \bar{i} \bar{\partial} \bar{\partial} = d^c d = -d d^c \tag{4.15}$$

where $d^c = Jd$. Now

$$L_{JX_H^{\omega_f}}(\omega_f) = d \circ i_{JX_H^{\omega_f}}(\omega_f)$$

and

$$\begin{aligned} i_{JX_H^{\omega_f}}(\omega_f)(Y) &= \omega_f(JX_H^{\omega_f}, Y) = \\ &= -\omega_f(X_H^{\omega_f}, JY) = -dH(JY) = -d^c H(Y). \end{aligned}$$

Therefore, indeed,

$$L_{JX_H^{\omega_f}}(\omega_f) = -d d^c H = z \bar{i} \bar{\partial} \bar{\partial} H \quad \blacksquare$$

Furthermore the orbit $\mathcal{F}((\omega, J))$ has a G_r -invariant metric, the Ma-

$$\langle h_1, h_2 \rangle_{\omega_f} = \int h_1 h_2 \frac{\omega_f^n}{n!} \quad (4.16)$$

M

Then, as expected from symmetric space metrics on K_C/K for compact Lie groups K , Mabuchi, Semmes and Donaldson prove the following very beautiful result for the curvature.

Thm 4.2 [Mabuchi - Semmes - Donaldson]

$$R_{\omega_f}(h_1, h_2) h_3 = -\frac{1}{4} \{ \{ h_1, h_2 \}_{\omega_f}, h_3 \}_{\omega_f} \quad (4.17)$$

and for the, non positive, sectional curvature

$$k_{\omega_f}(h_1, h_2) = -\frac{1}{4} \| \{ h_1, h_2 \}_{\omega_f} \|^2_{\omega_f} \quad (4.18)$$

□

Then, as in finite dimensions, we expect geodesics in G_C/G (K_C/K) to be given by imaginary time 1-parameter subgroups of G_C , which is why we have the three step approach to construct

$$\varphi_{is}^{X_H, J_0} \quad (4.19)$$

and we then prove that the path of metrics

$$\gamma_s = \omega(\cdot, (\varphi_{is}^{X_H, J_0})^*(J_0) \cdot) \quad (4.20)$$

is a MSD geodesic.

References for Lectures 1-3

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