

# Lecture 2

2/11/2015

2.1

## 3. Examples: Kähler tropicalization of $\mathbb{C}^*$ and $\mathbb{C}$ (lectures 2 and 3)

### Summary of this section

We will discuss here the first examples of Kähler tropicalization via special toric holomorphic discs in the infinite dimensional space of toric Kähler structures.

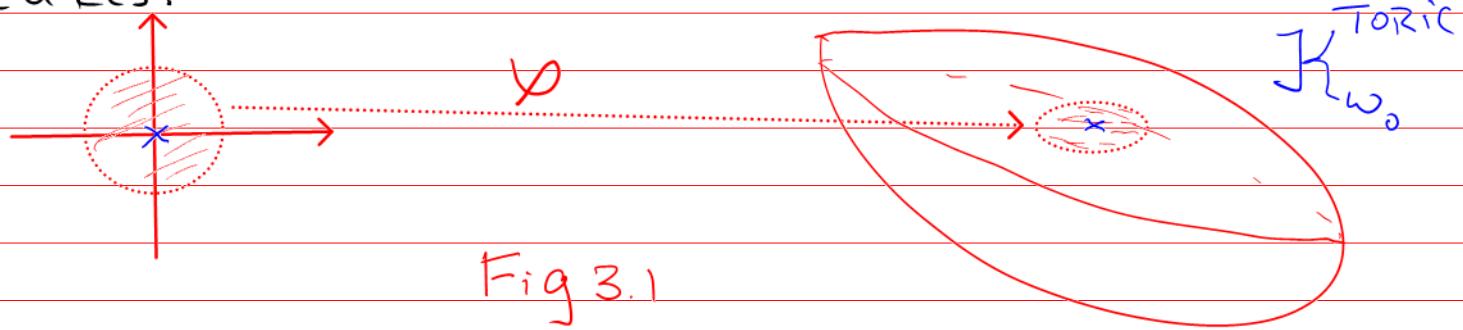


Fig 3.1

$$\varphi: e^{-s+i t} \mapsto \varphi_{t+is} \mapsto (\omega, J_{t+is}) = (\omega, (\varphi_{t+is})^* J_0) \quad (3.1)$$

As we will show later in the course to get a toric Kähler tropicalizing disc we need to choose as initial velocity  $k_0$  for the geodesic equation on the space of Kähler potentials a strictly convex function of the moment map components.

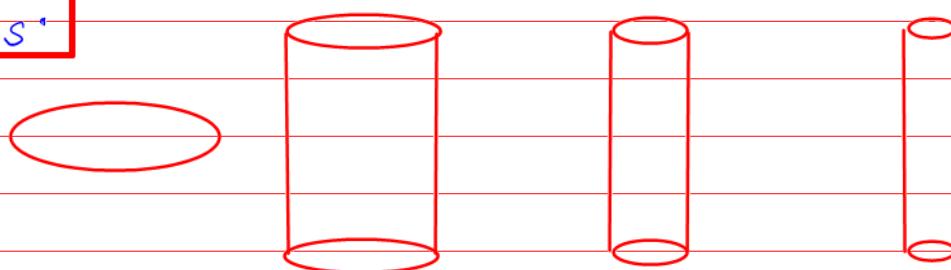
Let us summarize in pictures the tropicalizing geodesics (in the sense that the orbits of  $T^1$  are collapsing as  $s \rightarrow 0$ )

C\* - initial conditions:

$$w = e^{y+i\theta}$$

$$K_0(w) = \dot{K}_0(w) = \frac{y^2}{2} = \frac{(\log w)^2}{2} \quad (3.2)$$

$\gamma_s$ :



Moment  
Polyhedron  
 $\mu(C^*)$

$$s = -1$$

$$s=0 < s_1 < s_2 < s=\infty$$

Fig. 3.2a

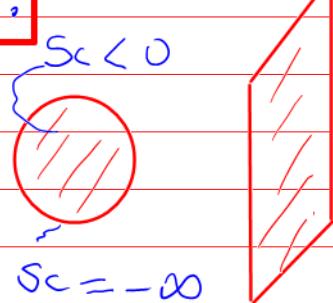
C

- initial conditions:

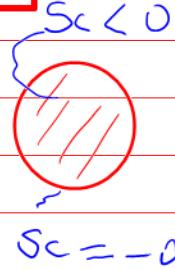
$$w$$

$$K_0(w) = \frac{|w|^2}{2}; \quad \dot{K}_0 = |w|^2 + \alpha \frac{|w|^4}{2}$$

$\gamma_s$ :



$$Sc = 0$$



$$Sc = -\infty$$

$$Sc > 0$$



Sc = Scalar curvature

$$Sc = \delta(x)$$

$$x = \begin{matrix} * \\ 0 \end{matrix} \parallel \mu(C)$$

$$s < 0 < s=0 < s_1 < s_2 < s=\infty$$

Fig. 3.2b

Recall that the method that we are using to find the geodesic satisfying the Cauchy problem

$$\left\{ \begin{array}{l} \ddot{k}_s = -\frac{1}{2} \|\nabla \dot{k}_s\|_{k_s}^2 \\ k_0 \\ \dot{K}_0(w) = H(w) \end{array} \right. \quad (3.3)$$

consists in finding the complexified symplectomorphisms,  $\varphi_{is}$ :

$$\begin{cases} \gamma_s = \omega(\cdot, J_{is}\cdot) \\ J_{is} = \varphi_{is}^*(J_0) \end{cases} \quad (3.4)$$

and these we obtain in 3 steps that just use the usual flow of  $H = K_0$ .

find  $\varphi_t^{X_H}$   
to  $J_{is}$

Step 1

Restrict  
to  $O_{J_0}$   
Define  
 $\Theta_{J_{is}} = (\varphi_t^{X_H})^* O_{J_{is}}$  (\*)

Step 2

Find unique  
diffeo  $\varphi_{is}^{X_H, J_0}$   
 $\Theta_J = (\varphi_{is}^{X_H, J_0})^* \Theta_{J_0}$

Step 3

### Example 3.1

Kähler tropicalization of  $\mathbb{P}^*$

$$w = e^z = e^{y+i\theta} \in \mathbb{C}^*$$

We choose  $K_0$  by choosing the natural flat metric on  $\mathbb{C}^*$

$$\gamma_0 = dy^2 + d\theta^2 \quad (3.5)$$

We can choose a  $T^1$ -invariant kähler potential, eg.

$$K_0(w) = \frac{y^2}{2} = \frac{(\log|w|)^2}{2} \quad (3.6)$$

Indeed,

$$\omega = 2i \partial \bar{\partial} K = \frac{2i}{2} \left( \frac{\partial}{\partial y} - i \frac{\partial}{\partial \theta} \right) \left( \frac{\partial}{\partial y} + i \frac{\partial}{\partial \theta} \right) \frac{y^2}{2} dz d\bar{z}$$

$$= \frac{\partial^2 K_0}{\partial y^2} dy \wedge d\theta = \frac{\partial^2}{\partial y^2} \left( \frac{y^2}{2} \right) dy \wedge d\theta = dy \wedge d\theta \quad (3.7)$$

and

$$\gamma_0 = dy^2 + d\theta^2 \quad (3.8)$$

From (3.7) we see that the Hamiltonian vector field of  $y$  is  $- \frac{\partial}{\partial \theta}$

$$X_y = - \frac{\partial}{\partial \theta} \quad (3.9)$$

which means that  $y$  coincides (at  $s=0$ ) with the moment map of the  $T'$ -action

$$\mu_0(\omega) = y \quad (3.10)$$

**Rmk 3.1**

As we will see the moment map of a toric kähler metric is the Legendre transform associated with the kähler potential (seen as a function on  $\text{Lie}(T^1)^* y$ )

$$x := \mu(\omega) = \frac{\partial K}{\partial y}, \quad (3.11)$$

where we used  $x$  for the moment map component. We therefore have

$$x = y = \frac{\partial}{\partial y} K_0 = \frac{\partial}{\partial y} \left( \frac{y^2}{2} \right) \quad (3.12)$$

As we start following a geodesic  $K_0 \mapsto K_s$  the relation between  $x$ :  $\omega = dx \wedge d\theta$  and  $y := \log \lvert w \rvert$

will change accordingly

2.5

$$\chi = \mu_{is}(y) = \frac{\partial K_{is}}{\partial y} \quad (3.13)$$

(see 3.18 below).  

Now for  $K_0$ , for reasons that will be explained later in the course we want a strictly convex function of  $\mu_0 = \chi$ . Let us choose

$$K_0(w) = H = \frac{\chi^2}{2} = k_0 \quad (3.14)$$

and run our algorithm of 3 steps to find  $\gamma_s$  and therefore also  $K_s$ .

Step 1 Find  $\varphi_t^{X_H}$  and  $t_m$  is

We have  $X_H = -\chi \frac{\partial}{\partial \theta}$  and therefore for its flow

$$\begin{cases} \dot{\Theta}_t = -\chi_t \\ \dot{\chi}_t = 0 \end{cases} \quad \begin{cases} \Theta_t = \Theta - t\chi \\ \chi_t = \chi \end{cases} \quad (3.15)$$

Then, on local real analytic functions, we can analytically continue in  $t$

$$\left( \varphi_t^{X_H} \right)^* : \quad \begin{cases} \Theta \mapsto \Theta - is\chi \\ \chi \mapsto \chi \end{cases} \quad (3.16)$$

Step 2 Restrict (3.16) to  $J_0$  holomorphic functions and (if possible) define  $J_{is}$

$$\left( \varphi_t^{X_H} \right)^* : w = e^{\chi + i\Theta} \mapsto e^{\chi + i(\Theta - is\chi)} = e^{(1+s)\chi + i\Theta} = w_{is} \quad (3.17)$$

So the question now is whether (3.17) defines a new complex structure  $J_s$  for which  $w_{is}$  is a  $J_s$ -holomorphic coordinate. The answer is yes for  $s \neq -1$ . We see already from (3.17) that

$$y_{is} = \log |w_{is}| = (1+s)x \quad (3.18)$$

**Step 3**

There is a unique diffeomorphism  $\psi \in \text{Diff}(\mathbb{C}^*)$  such that

$$\psi^*(w) = w_{is}, \quad (3.19)$$

which is, by definition 3 of Lecture 1,  $\psi_{is}^{X_{H1}, J_0}$ ,

$$\begin{aligned} \psi_{is}^{X_{H1}, J_0} : w = e^{x+i\theta} &\mapsto w_{is} = e^{x_{is} + i\theta_{is}} = \\ &= e^{(1+s)x + i\theta} \end{aligned} \quad (3.20)$$

and therefore our imaginary time flow is

$$\psi_{is}^{X_{H1}, J_0}(\theta, x) = (\theta, (1+s)x) \quad (3.21)$$

This concludes the 3 steps and we are now ready to find the geodesics  $\gamma_s$ :

$$\psi_{is}^{X_{H1}, J_0} \Rightarrow w_{is} = (\psi_{is}^{X_{H1}, J_0})^*(w) \Rightarrow J_s \Rightarrow \gamma_s = w(\cdot, J_s \cdot) \quad (3.22)$$

Short  
cut

**Rmk 3.2** Notice the peculiar way the imaginary time symplectomorphism (3.21) is acting on the metric  $J_0$  to produce  $J_s$  (let us call this action  $\Psi_s$  because we are fixing the symplectic structure. For more

details see Section 4)

Let  $\varphi_{is} := \varphi_{is}^{X_H, J_0}$ , then

$$\gamma_s = \rho_{sy}(\varphi_{is}^{-1})(\gamma_0) = \omega(\cdot, \varphi_{is}^* J_0 \cdot) \quad (3.23)$$

We see that the diffeomorphism  $\varphi_{is}$ , under  $\rho_{sy}$ , does not act on the symplectic form even though (3.22), as expected, is clearly not a symplectomorphism.

As we will see in the next section, the other non standard action of diffeomorphisms from  $G_\mathbb{C}$  is the complex picture action in which one leaves the complex structure untouched (even though  $\varphi_{is}$  does not preserve  $J_0$  either)

$$\tilde{\gamma}_s = \rho_{co}(\varphi_{is})(\gamma_0) = (\varphi_{is}^{-1})^*(\omega)(\cdot, J_0 \cdot) \quad (3.24)$$



Going back to finding  $\gamma_s$  notice that it could have been calculated after step 2 by finding  $J_{is}$ :

$$J_{is} = i \frac{\partial}{\partial w_{is}} \otimes dw_{is} - i \frac{\partial}{\partial \bar{w}_{is}} \otimes d\bar{w}_{is} \quad (3.25)$$

A more direct calculation (the shortcut indicated in (3.22)) is by expressing the symplectic form  $\omega$  in the (time dependent in the symplectic picture) holomorphic coordinates

$$\begin{aligned} \omega &= dx \wedge d\Theta = \gamma_{ii}^s \underbrace{dz_{is} \wedge d\bar{z}_{is}}_{-2i} = \\ &= \gamma_{ii}^s \underbrace{d((1+s)x + i\Theta) \wedge d((1+s)x - i\Theta)}_{-2i} = \end{aligned}$$

$$= \gamma_{\bar{z}}^s (1+s) dx \wedge d\theta \Leftrightarrow$$

$$\Leftrightarrow dx \wedge d\theta = \gamma_{\bar{z}}^s (1+s) dx \wedge d\theta \Leftrightarrow$$

$$\Leftrightarrow \gamma_{\bar{z}}^s = \frac{1}{1+s} \quad (3.26)$$

Then,

$$\gamma_s = \frac{1}{1+s} dz_{is} \otimes d\bar{z}_{is} =$$

$$= \frac{1}{1+s} ((1+s)^2 dx^2 + d\theta^2) \Leftrightarrow$$

$$\boxed{\gamma_s = (1+s) dx^2 + \frac{1}{1+s} d\theta^2}, \quad (3.27)$$

in full agreement with Fig. 3.2a.