

Lecture 1

S1 Introduction

Let (M, ω) be a symplectic manifold, $H \in C^\infty(M)$, X_H be the Hamiltonian vector field of H and $\varphi_t^{X_H}$ be the flow of X_H , i.e.

$$\varphi_t^{X_H} \in \text{SDiff}_0(M, \omega) =: G \quad (1.1)$$

and $x_t = \varphi_t^{X_H}(x)$ are solutions of

$$\begin{cases} \dot{x}_t = X_H(x_t) \\ x_0 = x \end{cases} \quad (1.2)$$

The goal of this minicourse will be to give meaning to the analytic continuation of $\varphi_t^{X_H}$ in $t \mapsto \tau \in \mathbb{C}$ but in such a way that the result is still a diffeomorphism of M

$$\varphi_t^{X_H} \rightsquigarrow \varphi_\tau^{X_H} \in \text{Diff}(M) \quad \tau \in \mathbb{C} \quad (1.3)$$

and to discuss some applications of this formalism.

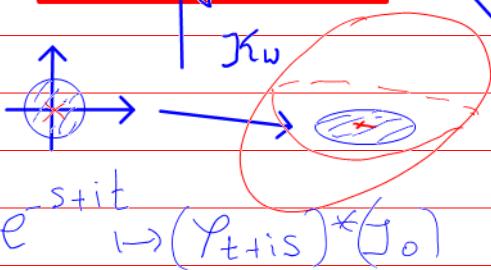
In Kähler geometry:

construct geodesics in the infinite dimensional space of Kähler structures K_ω on M

Applications of complex time flows

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In tropical geometry:

By choosing appropriate K_ω construct holomorphic geodesic disks in K_ω which tropicalize (see figure)

1.2

B

In Quantum Physics:

Study (non)equivalence of different quantizations

$$U_\zeta : \mathcal{H}_{P_0} \longrightarrow \mathcal{H}_{P_1}$$

B

Study quantization in singular real representations

In representation theory:

Find special basis in spaces of holomorphic sections

Give a geometric interpretation to Peter-Weyl like theorems

Provide a new perspective on symplectic Reduction

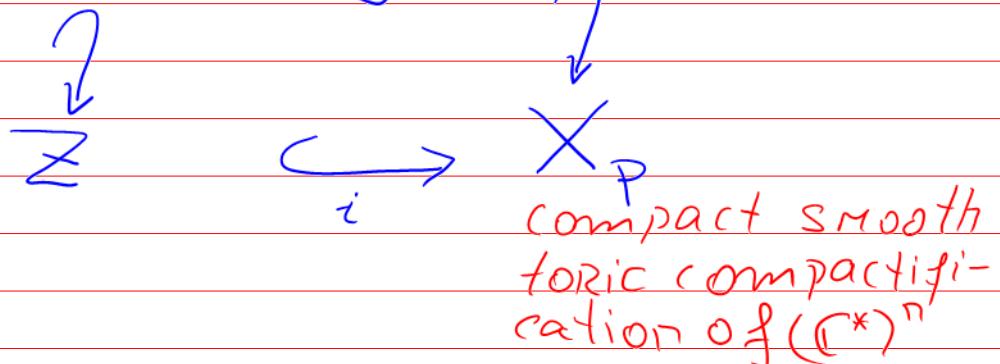
d: To be discussed in the minicourse

B: Not to be discussed

Illustration Tropical Geometry

Let us summarize what the main claim is in tropical geometry. Let \tilde{Z} be an algebraic hypersurface in $(\mathbb{C}^*)^n$ with Newton polytope P which we assume is Delzant

$$\tilde{Z} = \left\{ \sum_{m \in \mathbb{Z}^n \cap P} a_m w^m = 0 \right\} \subset (\mathbb{C}^*)^n$$



Next we want to find a tropicalizing geodesic disk of toric Kähler structures on X_P .

For this we need to choose appropriate initial conditions for the geodesic disk namely a initial Kähler potential k_0 (any toric (ie. T^n -invariant)) adapted to P and a initial velocity \dot{k}_0 . This we choose global

$$\dot{k}_0 = H \in C^\infty(M), \quad (1.4)$$

toric and strictly convex as function of the toric periodic hamiltonians.

$$\dot{k}_0(w) = H(w) = \tilde{H}(x_1, \dots, x_n) \quad (1.5)$$

Then if k_s denotes the solution of the geodesic equations with these initial conditions let μ_s be the corresponding moment map at geodesic time s [the Hamiltonian time is $\tau = \sqrt{-1}s$]

$$\mu_s(w) = \left(\frac{\partial k_s(\|w\|)}{\partial y_1}, \dots, \frac{\partial k_s(\|w\|)}{\partial y_r} \right) \quad (1.6)$$

$$w = e^{yt+i\theta} = (e^{y_1+it\theta}, \dots, e^{y_n+it\theta})$$

Compact tropical amoebas at geodesic time s are

$$A_s^c = \mu_s(z) \quad (1.7)$$

and the claim is that they tropicalize as $s \rightarrow \infty$ as we will see later in the minicourse.

§2 Definitions of complex time Hamiltonian flows and some results

Historical diagram for the complex time Hamiltonian flows

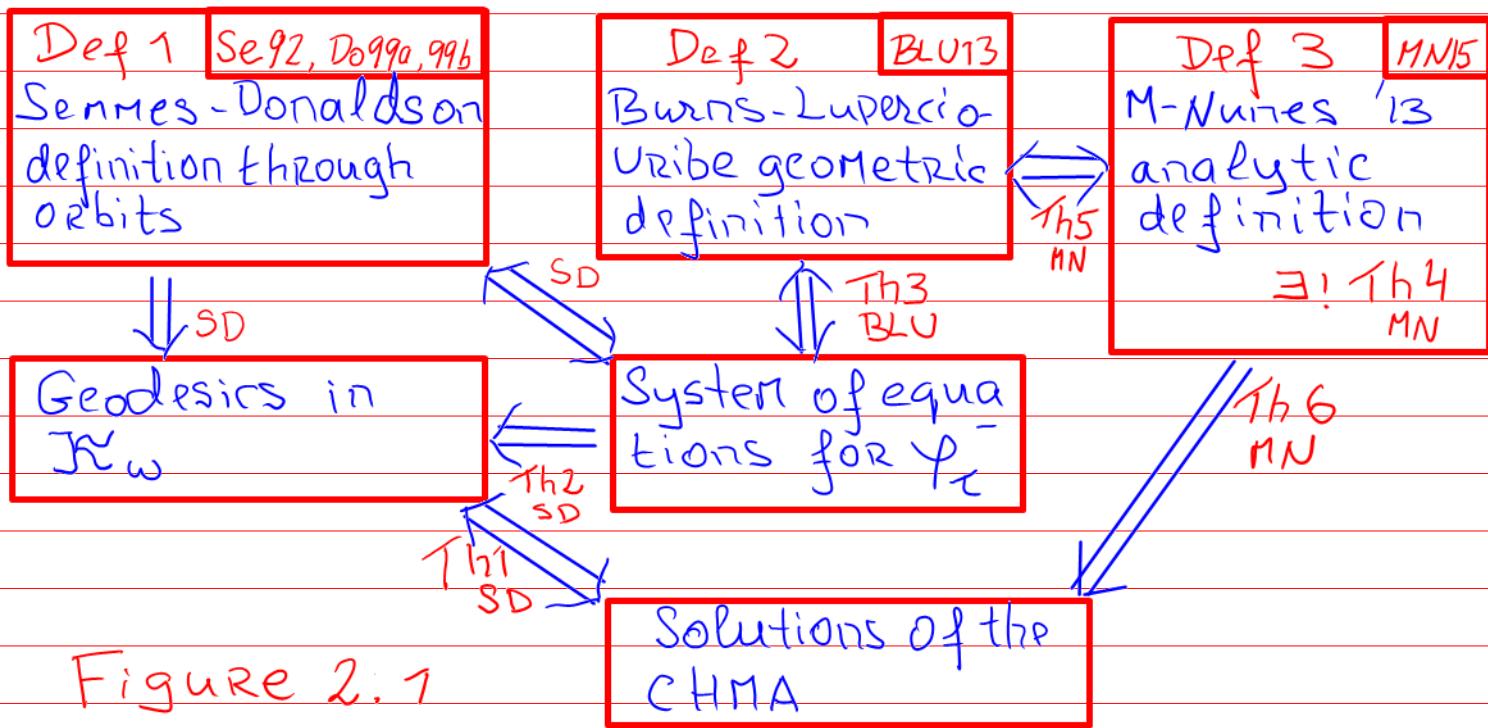


Figure 2.1

Let us now introduce definitions 2 and 3 and study the theorems 3-5. We will return to theorems 1, 2 later in the course

PROP 2.1 ([BLU13] PROP 1.1])

Let (M^{2n}, ω, J_0) be a real analytic Kähler manifold. There exists a holomorphic symplectic complex manifold (M_C, ω_C, J_C) with $\dim_{\mathbb{C}}(M_C) = 2n$ with embedding $i: M_C \hookrightarrow M$, s.t.

$$i^* \omega_C = \omega \quad (2.1)$$

and with the following additional structures

1. an anti-holomorphic involution

$\tau: M_{\mathbb{C}} \rightarrow M_{\mathbb{C}}$, such that $\tau^* \omega_{\mathbb{C}} = \overline{\omega}_{\mathbb{C}}$

2. a holomorphic projection $\pi: M_{\mathbb{C}} \rightarrow M$ whose fibers are holomorphic Lagrangian submanifolds.

Proof Local existence: take a nbhd of the diagonal in $M \times M$ with complex structure $I = (-, -)$ and M embedded in the diagonal. Choose as $\omega_{\mathbb{C}}$ the holomorphic extension of ω

$$\omega = \sum g_{j\bar{k}}(z, \bar{z}) \frac{dz_j \wedge d\bar{z}_k}{-z_i} \quad (2.2)$$

$$\omega_{\mathbb{C}} = \sum g_{j\bar{k}}(z, w) \frac{dz_j \wedge dw_k}{-z_i}$$

see Fig. 2.1

and $\pi_*(z, w) = z$.

Let $H \in C^\omega(M, \mathbb{C})$, $H = H_1 + iH_2$

Def 2 [of $\varphi_t^{X_H}$] (Geometric def. [BLU13])

Let the fibers of π_* be $F = \{F_x = \pi_*^{-1}(\{x\})\}_{x \in M}$ and $\phi_t: M_{\mathbb{C}} \rightarrow M_{\mathbb{C}}$ be the Hamiltonian flow of $-\text{Re}(H_{\mathbb{C}}) = -\text{Re}H_{\mathbb{C}} + i\text{Im}H_{\mathbb{C}}$ with respect to $\text{Re}(\omega_{\mathbb{C}})$ on $M_{\mathbb{C}}$, i.e.

$$X_{-\text{Re}H_{\mathbb{C}}} = X_{H_{\mathbb{C}}}^{\omega_{\mathbb{C}}} - X_{\overline{H}_{\mathbb{C}}}^{\overline{\omega}_{\mathbb{C}}} \quad (2.3)$$

Let F_t be the image of the fibration F_0 under $(\phi_t)^*$. Assume there exists $\epsilon > 0$ such that $\forall t: |t| < \epsilon$, $(F_t)_{z \in F_0}$ are the leaves of a fibration

with projection $\Pi_t : M_C \rightarrow M$. Then define

$$\begin{array}{ccc} M_C & \xrightarrow{\Phi_t} & M_C \\ i \uparrow & & \downarrow \Pi_t \quad \text{i.e.} \\ M & \xrightarrow{\phi_t} & M \end{array} \quad (2.4)$$

$$\phi_t = \Pi_t \circ \Phi_t \circ i$$

□

See Fig 2.1

Theorem 3 (Th 1.2 of [BLU13])

(1) $\forall t \in (-\epsilon, \epsilon)$ \exists complex structure J_t such that Π_t is holomorphic

$$(\Pi_t)_* \circ J_0 = J_t \circ (\Pi_t)_* \quad (2.5)$$

(2) $\phi_t : (M, J_0)$ $\rightarrow (M, J_t)$ is holomorphic

$$(\phi_t)_* \circ J_0 = J_t \circ (\phi_t)_* \quad (2.6)$$

(3) The infinitesimal generator of ϕ_t is

$$\begin{aligned} \phi_t \circ \phi_t^{-1} &= -X_{H_1} - J_t X_{H_2} = \\ &= -X_{H_1} - \nabla^J H_2 \end{aligned} \quad (2.7)$$

□

We see that to the imaginary part of H there corresponds a gradient motion with a metric which is itself being changed by the flow

$$\gamma_t = \omega(\cdot, (\phi_t)_*(J_0) \cdot) \quad (2.8)$$

so that (2.6) and (2.7) form a coupled system of equations. By substitu-

ting (2.6) (or (2.8)) into (2.7) we get
a linear PDE for ϕ_t .

Let us now describe the analytic approach in which one only needs to solve the ODE for the complexified integral curves of X_H to define $\psi_t^{X_H}$. These are formal integral curves as what is in fact being defined is a local automorphism of $C^w(M, \mathbb{C})$, the algebra of real analytic complex valued functions on M ,

$$C^w(M, \mathbb{C}) \ni f \mapsto (\psi_t^{X_H})^*(f) \quad (2.9)$$

In particular, if X_H has non zero imaginary part then some real valued functions turn to complex valued functions, thus ruining any hope that there could be any diffeomorphism having the same effect as (2.9). Of course one needs somehow to project back to M like in def. 2. Our way to project back is to choose a complex structure \mathcal{J}_0 on M and **restrict** the action of $\psi_t^{X_H}$ to local on open sets $U \subset M$, \mathcal{J}_0 -holomorphic functions

$$\Omega_{\mathcal{J}_0} \ni f_0 \mapsto (\psi_t^{X_H})^*(f_0) \quad (2.10)$$

It turns out that, by doing this for t sufficiently small and M compact, there is a unique \mathcal{J}_0 -dependent diffeomorphism of M , which we denote by $(\psi_t^{X_H, \mathcal{J}_0})$, such that

$$(\psi_t^{X_H, \mathcal{J}_0})^*(f_0) = (\psi_t^{X_H})^*(f_0) \quad (2.11)$$

for all local \mathcal{J}_0 -holomorphic functions f_0

on sufficiently small sets. The resulting functions $\tilde{\psi}_t = (\psi_{t+J_0}^{X_{H_1}})^*(f_0)$ define a unique complex structure \tilde{J}_t for which $\tilde{\psi}_t$ is holomorphic. Let us state this more precisely and sketch the proof of existence and uniqueness of $\psi_{t+J_0}^{X_{H_1}}$. To simplify the exposition we will consider the case when

$$H = \tilde{H} = \underbrace{\tau_1 \tilde{H}}_{H_1} + i \underbrace{\tau_2 \tilde{H}}_{H_2} \quad (2.12)$$

with \tilde{H} real valued, so that we can speak of real time flow of $X_{\tilde{H}}$ being analytically continued to complex time

$$\psi_t^{X_H} \rightsquigarrow \psi_t^{X_{H,J_0}} \quad (2.13)$$

using the same steps as above. We will then have

$$\psi_t^{X_{H,J_0}} = \psi_t^{X_{\tilde{H}, J_0}} = \psi_{t\tau}^{X_{\tilde{H}}, J_0} \quad (2.14)$$

The analytic approach can be extended to general analytic complex valued Hamiltonians by using exponential series of vector fields,

$$\psi_t^X = \sum_{k=0}^{\infty} \frac{t^k}{k!} X^k \quad (2.15)$$

a theory developed by Gröbner and co-workers and called Gröbner theory of Lie series. For more details see [MN15].

Def 3 (analytic [MN15]). Let $H \in C^0(M, \mathbb{R})$.

$\psi_t^{X_H, J_0}$ is defined through the following 3 steps.

S1 Let X_H be a complete vector field and let $\varphi_t^{X_H}$ denote its real time flow. Act by pullback on real analytic (local) functions and analytically continue in time
 $f \mapsto (\varphi_t^{X_H})^*(f)|_{t \mapsto \tau} \quad (2.16)$

S2 Choose a complex structure J_0 and restrict the action (2.16) to local J_0 -holomorphic functions
 $O_{J_0}(U) \ni f_0 \mapsto (\varphi_t^{X_H})^*(f_0)|_{t \mapsto \tau} = f_\tau \quad (2.17)$

Suppose that there exists a complex structure J_τ on M such that f_τ are J_τ -holomorphic.

S3 If f_τ exists then define $\psi_t^{X_H, J_0}$ as the (unique*) diffeomorphism of M such that

$$(\psi_t^{X_H, J_0})^*(f_0) = \varphi_t^{X_H}(f_0)|_{t \mapsto \tau} \quad (2.18)$$

for all local J_0 -holomorphic functions for which the rhs of (2.18) is well defined.

* see thm 4

Thm 4

1.10

existence and uniqueness of $\varphi_{t^H, J_0}^{X_H}$
 Thus 2.6 and 3.1 and
 sections 7 and 8 of [HN15]

Let (M, ω, J_0) be a compact real analytic Kähler manifold and $H \in C^\omega(M, \mathbb{R})$. Then $\exists T_H$, $0 < T_H \leq \tilde{T}_H$ such that $\varphi_{t^H, J_0}^{X_H}$ and J_t^H as in def 3 exist and are unique, $t \in \mathbb{C}$: $|t| < T_H$. For all t : $|t| < \tilde{T}_H$ there exists a unique map $\varphi_t^{X_H, J_0}$,

$$\varphi_t^{X_H, J_0} : M \rightarrow M \quad (2.19)$$

defined by (2.18) but it may not be a diffeomorphism. If $\varphi_t^{X_H, J_0}$ is not a diffeomorphism then

$$(\varphi_t^{X_H, J_0})^* J_0 \quad (2.20)$$

is not a complex structure and defines a mixed polarization (in the sense of geometric quantization; see examples in the next section)

Proof (sketch)

Let z_j be local J_0 -holomorphic coordinates. We define

$$z_j^t := (\varphi_t^{X_H})^*(z_j) /_{t \mapsto t} \quad (2.21)$$

For $|t|$ small enough and compact M , z_j^t define a complex coordinate system and a complex structure J_t such that z_j^t are J_t -holomorphic. In fact, given J the automorphism properties

1.11

of $\Psi_t^{X_H}$ the transition functions "don't change", i.e.

$$z^{(\beta)} = F^{\beta\alpha}(z^{(\alpha)}) \quad (2.22)$$

$$\left. \begin{array}{c} \\ \downarrow \\ z_t^{(\beta)} = F^{\beta\alpha}(z_t^{(\alpha)}) \end{array} \right\}$$

and therefore (M, J_0) and (M, J_T) are biholomorphic with biholomorphism defined by:

$$\Psi_t^{X_H, J_0}(z, \bar{z}) := (z_t(z, \bar{z}), \overline{z_t(z, \bar{z})}), \quad (2.23)$$

where

$$z_t(z, \bar{z}) = (\Psi_t^{X_H})^*(z) \Big|_{z=\bar{z}}$$

Thm 5 (equivalence of Def 2 and Def 3;
PROP 8.1 of [MN15])

Let $H = \tau \tilde{H} = \tau_1 \tilde{H} + i \tau_2 \tilde{H}$

Then

$$\underbrace{\Psi_{t\tau}^{X_H, J_0}}_{\text{Def 3}} = \underbrace{(\phi_t)^{-1}}_{\text{Def 2}} \quad (2.24)$$

Def 3 Def 2

i.e. the following diagram is commutative

$$\begin{array}{ccc} (M_C, J) & \xrightarrow{\Phi_t} & (M_C, J) \\ i \uparrow & & \downarrow \pi_t \\ (M, J_0) & \xleftrightarrow{\Psi_{t\tau}^{X_H, J_0}} & (M, J_{t\tau}) \end{array} \quad (2.25)$$

Proof (sketch)

Recall from def 2 that $\Phi_t = \text{flow of } -\text{Re}(H_C)$, where H_C is the analytic continuation of H to M_C (see (2.3))

$$X_{-\text{Re } H_C}^{\text{Re } \omega_C} = -X_{H_C}^{\omega_C} - X_{\bar{H}_C}^{\bar{\omega}_C}$$

so that

$$\left. X_{-\text{Re } H_C}^{\text{Re } \omega_C} \right|_{\substack{\text{I-holom} \\ \text{functions}}} = \left. X_{-H_C}^{\omega_C} \right|_{\substack{\text{I-holo} \\ \text{morphi} \\ \text{functions}}} \quad (2.26)$$

Let f be a (local) $J_{t\bar{t}}$ holomorphic function. Then

$$(\phi_t^*)(f)(z, \bar{z}) = i^* \circ \phi_t^* \circ \eta_t^*(f)(z, \bar{z})$$

$$\begin{cases} \eta_t^*(f)(z, w) = f(z_{t\bar{z}}) \\ \phi_t^*(f)(z, w) = f(z) \\ i^*(f)(z, \bar{z}) = f(z). \end{cases} \quad (2.27)$$

We see that ϕ_t maps $J_{t\bar{t}}$ -holomorphic functions to $J_{t\bar{t}}$ -holomorphic functions and coincides with $(\varphi_{X_{H_C}})^{-1}$ since it maps $z_{t\bar{t}}$ to z . (See (2.21) and (2.23)).

References for Lecture 1

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