Imaginary time in Kähler geometry, quantization and tropical amoebas

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1. Introduction to dynamics in Imaginary time and Main Result

This seminar and the minicourse that will follow is very much about Hamiltonian systems

$$(M, \omega, H)$$

that is symplectic manifolds (M, ω) with a chosen function $H \in C^\infty(M)$ and the associated Hamiltonian flow

$$\varphi_t^{X_H} = \text{"}e^{tX_H}\text{"} \in SDiff_0(M,\omega) \subset Diff(M) \qquad (1.1)$$

where X_H is the Hamiltonian vector field of H,

$$X_H : \quad dH = \omega(X_H, \cdot) \qquad (1.2)$$

The novelty is that for the sake of the applications mentioned in the title – Kähler geometry, quantization and tropical amoebas – we will be led to try to make sense of the analytic continuation of the flow (1.1) to complex time

$$\varphi_\tau^{X_H}, \tau \in \mathbb{C} \qquad (1.3)$$

This would not be a big deal if the resulting flow acted on a complexification $M_\mathbb{C}$ of M

$$\varphi_\tau^{X_H} \overset{?}{\in} Diff(M_\mathbb{C}) \qquad (1.4)$$

It turns out however that for the applications we need

$$\varphi_\tau^{X_H} \underset{??}{\in} \mathrm{Diff}(M) \cup C^\infty(M,M) \cup \dots \qquad (1.5)$$

We will see that to define $\varphi_\tau^{X_H}$ we will have to fix an additional initial geometric structure that can be a complex structure J_0 compatible with the symplectic form

$$\omega(J_0 X, J_0 Y) = \omega(X, Y) \qquad (1.6)$$
$$\forall\, X, Y \in \mathfrak{X}(M)$$

to make a pseudo-Kähler pair (ω, J_0)

To get an idea of what is motion in complex time let me write the equations satisfied by the following real 1-parameter family:

fix $\tau \in \mathbb{C}$, $H_0 \in C^\infty(\mathbb{R})$ and consider

$$\varphi_{t\tau}^{X_{H_0}, J_0} = \varphi_t^{X_{\tau H_0}, J_0}, \quad t \in \mathbb{R} \qquad (1.7)$$

So we are interested in the real time-flow but with respect to the complex Hamiltonian

$$H = \tau H_0 = \tau_1 H_0 + i\, \tau_2 H_0$$

Def. 1.1 [Semmes'92, Donaldson '99, Burns-Lupercio-Uribe '13]

Let (M, ω, J_0) be a compact Kähler manifold and $H \in C^\omega(M, \mathbb{C})$. Then $x_t = \left(\varphi_t^{X_H, J_0}\right)^{-1}(x_0)$

is solution of the following system

$X_{H_1} + i X_{H_2}$:

See [Ch00, BLU+13, CS14, Ze15]

$$\begin{cases} \boxed{\begin{aligned} \dot{x}_t &= -X_{H_1}(x_t) - J_t X_{H_2}(x_t) = \\ &= -X_{H_1} - \nabla^{\gamma_t} H_2 \end{aligned}} \\ \text{where } J_t = \left(\varphi_t^{X_{H_1}, J_0}\right)^* J_0 \qquad (1.8) \\ \text{and } \gamma_t = \omega(\cdot, J_t \cdot) \end{cases}$$ $\square$

We see that the imaginary part of H gives a gradient motion but for a changing metric. Let us illustrate with the simplest non trivial case

$\boxed{\text{Ex. 1.2}}$ $(M, \omega) = (\mathbb{R}^2, dx \wedge dp)$; $H = p^2/2$

J_0 : $z = x + ip$; $J_0 \Leftrightarrow \langle \frac{\partial}{\partial \bar{z}} \rangle$

	Vector field	Equations	Flow
Real time Hamiltonian	$X_H = p\frac{\partial}{\partial x}$	$\begin{cases} \dot{x}_t = p_t \\ \dot{p}_t = 0 \end{cases}$	$\begin{cases} x_t = x + pt \\ p_t = p \end{cases}$
Real time gradient	$\nabla^{\gamma_0} H = p\frac{\partial}{\partial p}$	$\begin{cases} \dot{x}_t = 0 \\ \dot{p}_t = p \end{cases}$	$\begin{cases} x_t = x \\ p_t = e^t p \end{cases}$
Imaginary time Ham. = Real time gradient with chang. metric	$\nabla^{\gamma_t} H = \frac{p}{1+t}\frac{\partial}{\partial p}$ $\wr$ iX_H	$\begin{cases} \dot{x}_t = 0 \\ \dot{p}_t = -\frac{p_t}{t+1} \end{cases}$	$\begin{cases} x_t = x \\ p_t = \frac{p}{t+1} \end{cases}$

Table 1.1

The changing metric is

$$\boxed{\gamma_{it} = \frac{dx^2}{1+t} + (1+t)\, dp^2 \qquad (1.9)}$$

We therefore have

$$\varphi_{it}^{X_H, J_0}(x,p) = \begin{pmatrix} 1 & 0 \\ 0 & 1+t \end{pmatrix} \begin{pmatrix} x \\ p \end{pmatrix} \qquad (1.10)$$

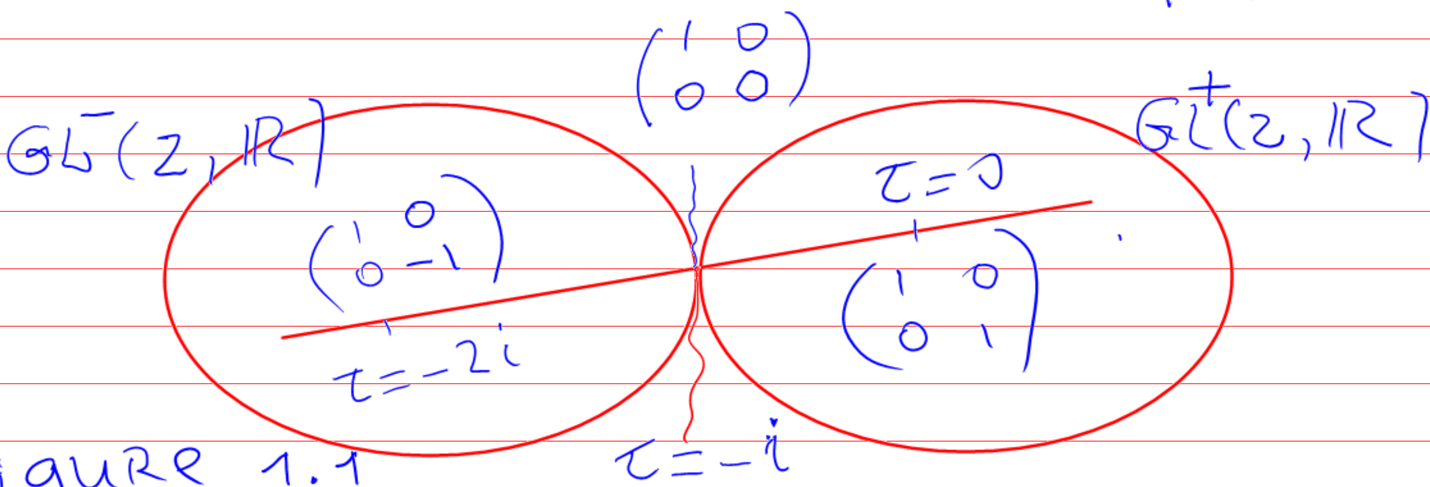

Figure 1.1

$$\varphi_{-i}^{X_H, J_0} \notin GL(2,\mathbb{R})$$

The coupled system (1.8) looks very difficult to solve.

The main contribution of [MN15] was to show that one can solve (1.8) by the following 3 steps

$\boxed{\text{Thm 1.3}}$ [M-Nunes] Let (M, ω, J_0) be a compact Kähler manifold. Then,

(S1) find the (local) flow of the time independent complex vector field $X_H = X_{H_1} + i X_{H_2}$ acting on $C^\infty(M, \mathbb{C})$

$$e^{tX_H} \circlearrowright C^\omega(M,\mathbb{C}) \;;\; |t| < R_H$$

$\boxed{\text{RMK}}$ If $H = (\tau_1 + i\tau_2) H_0$ then the above is equivalent to finding the Real time flow of X_{H_0}

$$\varphi_t^{X_{H_0}} \qquad (1.11)$$

and then analytically continue in time (as a local automorphism of $C^\omega(M,\mathbb{C})$) to

$$\left(\varphi_{t'}^{X_{H_0}}\right)^* \Big|_{t' = t(\tau_1 + i\tau_2)} (f) \quad \text{for } |t\tau| < R_{H,f} \qquad (1.12)$$

(S2) Restrict the flow e^{tX_H} to the local J_0-holomorphic functions and define

$$\mathcal{O}_{J_t} := \left(e^{tX_H}\right)_* \mathcal{O}_{J_0} \qquad (1.13)$$

(S3) Find the unique (for $|t| < R_H$; see Thm 1.3 below) diffeomorphism $\varphi_t^{X_H, J_0} \in \mathrm{Diff}(M)$ such that

$$\left(\varphi_t^{X_H, J_0}\right)^* \mathcal{O}_{J_0} = \left(e^{tX_H}\right)^* \mathcal{O}_{J_0} \qquad (1.14)$$

Then the pair

$$\boxed{\left(\left(\varphi_t^{X_H, J_0}\right)^{-1}(x), \left(\varphi_t^{X_H, J_0}\right)^*(J_0)\right)}$$

Solves (1.8) $\square$

Ex 1.4 Let us use thm 1.3 to find the solution of ex. 1.2 listed in the last row of the table and in (1.9).

So we have $(M, \omega) = (\mathbb{R}^2, dx \wedge dp)$, J_0 : $z = x + ip$ is J_0 - holomorphic and $H = p^2/2$.

We now use the steps of thm 1.3 [M is noncompact but the thm applies in this example]

S1 [see Rmk]
Find real time flow of $X_H = p\frac{\partial}{\partial x}$

$$\begin{cases} \left(e^{sX_H}\right)^*(x) = x + sp \\ \left(e^{sX_H}\right)^*(p) = p \end{cases} \quad (1.15)$$

+ analytically continue $s \rightsquigarrow it$

$$\begin{cases} \left(e^{isX_H}\right)^*(x) = x + itp \\ \left(e^{isX_H}\right)^*(p) = p \end{cases} \quad (1.16)$$

S2 Restrict $\left(e^{itX_H}\right)^*$ to J_0-holomorphic functions and define

$$J_{it} : \mathcal{O}_{J_{it}} := \left(e^{itX_H}\right)^* \left(\mathcal{O}_{J_0}\right)$$

So $z_{it} = \left(e^{itX_H}\right)^*(x + ip) =$

$$= x + i(1+t)p \quad (1.17)$$

S3 Find unique diffeo $\varphi_{it}^{X_{H},J_0}$:

$$\left(\varphi_{it}^{X_H,J_0}\right)^*(x+ip) = \left(e^{itX_{H}}\right)^*(x+ip)$$
$$= x + i(1+t)p =: z_{it} \qquad (1.18)$$

So indeed we get

$$\varphi_{it}^{X_H,J_0}(x,p) = (x, (1+t)p) \qquad (1.19)$$

as in the last row of the table

Let us verify one of the components of the metric in (1.9).

$$\gamma_{it}\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial x}\right) = \omega\left(\frac{\partial}{\partial x}, J_{it}\frac{\partial}{\partial x}\right) =$$

$$= \omega\left(\frac{\partial}{\partial x}, \frac{1}{1+t}\frac{\partial}{\partial p}\right) = \frac{1}{1+t} \qquad (1.20)$$

in agreement with (1.9) and with the system (1.8).

Rmk 1.5 (a) We see that thm 1.3 simplifies theorem 1.1 in the sense that it reduces the complicated coupled system (1.8) to find complex symplectomorphisms to finding the flow of the original time independent complex valued vector field on $C^\omega(M,\mathbb{C})$

(b) It opens the door to considering values of τ for which $\varphi_\tau^{X_H,J_0} \notin \mathrm{Diff}(M)$ like in figure 1.1

2 Imaginary time in geometry

2.1 App. 1: Constructing geodesics in the infinite dimensional space of Kähler metrics

[GS91, LS91, Se92, Th96, Do99a, Do99b, Ch00, RZ11, HK11, LS12, KMN13, BLU13, CS14, KMN15, MN15, Ze15]

Let $G = SDiff_0(M,\omega)$ denote the group of Hamiltonian symplectomorphisms of (M,ω).
In the previous section we defined "1-parameter" subgroups of the non-existing complexification $G_{\mathbb{C}}$ of G. We followed Donaldson's pragmatic approach to define $G_{\mathbb{C}}$ through its "orbits" on the spaces of (almost) complex structures and Kähler forms [Do99].

We will be, in the present section, spetially interested in the latter case.

Let (M,ω,J_0) be a compact Kähler manifold and consider the space $\mathcal{H}$ of Kähler potentials with Kähler form in the same cohomology class as ω.

$$\mathcal{H} = \{\phi \in C^\infty(M) : \omega_0 + i\bar\partial\partial\phi > 0\} \quad (2.1)$$

The space of Kähler forms with cohomology class $[\omega_0]$ is then

$$\mathcal{H}_0 \cong \mathcal{H}/\mathbb{R} \tag{2.2}$$

We see from (2.1) that $\mathcal{H}$ is an open subset of $C^\infty(M)$ so that we can identify its tangent space at any point $\phi \in \mathcal{H}$ with $C^\infty(M)$. $\mathcal{H}$ has a natural L^2-Riemannian metric

$$\langle f, g\rangle_\phi = \int_M f g \frac{\omega_\phi^n}{n!}, \quad f, g \in T_\phi \mathcal{H}$$

$\omega_\phi = \omega + i\partial\bar\partial\phi$ (2.3)

called Mabuchi-Semmes-Donaldson (MSD) metric. The geodesic equation is the following nonlinear EDP

$$\phi(t)'' = \frac{1}{2} |\nabla \phi'|^2_{\phi(t)} \tag{2.4}$$

equivalent to the homogeneous Monge-Ampère eq in n $n+1$ dimensions.

Why is $\mathcal{H}$ the analogy of $\tilde{G}_{\mathbb{C}}/\tilde{G}$?

Thm 2.1 (Donaldson '99)

$\mathcal{H}$ with the MSD metric (2.3) is an infinite dimensional symmetric space with curvature tensor (2.5)

$$R_\phi(f_1, f_2) f_3 = -\frac{1}{4}\{\{f_1, f_2\}_\phi, f_3\}_\phi$$

and therefore non-positive sectional curvature

$$K_\phi(f_1, f_2) = -\frac{1}{4}\|\{f_1, f_2\}\|^2 \tag{2.6}$$

$\square$

Formulas (2.5) and (2.6) are the full analogs of the formulas valid for the finite dimensional symmetric spaces

$$K_{\mathbb{C}}/K$$

for a compact Lie group K with $K_{\mathbb{C}}$ - invariant metric.

Furthermore, as in the finite dimensional case, the imaginary time "one-parameter subgroups" project to geodesics on the symmetric space

$$e^{it\xi} \mapsto e^{it\xi}K \in K_{\mathbb{C}}/K, \; \xi \in \mathrm{Lie}(K) \quad (2.7)$$

$$e^{itX_H} \longmapsto \left(\varphi_{it}^{X_H, J_0}\right)^*(\omega) \in \mathcal{H}_0, \quad (2.8)$$

$$\mathcal{H} = \mathcal{H}_0 \oplus \mathbb{R}$$

where

$$\mathcal{H}_0 = \{\omega_\phi = \omega + i\partial\bar\partial\phi, \phi : \omega_\phi > 0\} =$$

$$\underset{\text{Moser}}{=} \{\psi^*\omega, \psi \in \mathrm{Diff}(M) : (\psi^*\omega, J_0) \text{ is Kähler}\} \quad (2.9)$$

$$=: G_{\mathbb{C}}\,\omega$$

As mentioned above $G_{\mathbb{C}}$ appears defined through its "orbits" as (ω, J_0) - dependent subsets (not subgroups) of $\mathrm{Diff}(M)$.

Therefore the theorem 1.3 gives an efficient way of reducing the Cauchy problem for geodesics in $\mathcal{H}$ (the nonlinear PDE (2.14)) to finding

$$\varphi_{it}^{X_H} \rightsquigarrow \left(\varphi_{it}^{X_H}\right)^*(\omega) \qquad (2.10)$$

2.2 $\boxed{\text{Application 2:}}$ Special geodesics and tropical amoebas

[GKZ94, Gu94, Mi00, Ab10, B11, Ka11, IM12, KMN15, FMN15]

In this section we show how special geodesics in the space of Kähler toric potentials $\mathcal{H}$ allow to give a symplectic/Kähler interpretation of the $\boxed{\mathrm{Log}_t}$ map in amoeba theory with

$$\boxed{t = e^{s}} \qquad (2.11)$$

where s is the geodesic, or equivalently, the imaginary Hamiltonian time. This interpretation extends to the relevant compact toric variety (i.e. "compactifies").

Consider a complex hypersurface

$$\check{Z} = \Big\{ w \in \mathbb{C}^{*n} : \sum_{m \in P \cap \mathbb{Z}^n} a_m w^m = 0 \Big\} \subset (\mathbb{C}^*)^n \qquad (2.12)$$

If the Newton polytope P is Delzant then there is a smooth complex toric variety X_P to which (2.12) extends

$$\begin{tikzcd} Z \arrow[r, hook] & X_P \\ \check{Z} \arrow[u, hook] \arrow[r, hook] & (\mathbb{C}^*)^n \arrow[u, hook] \end{tikzcd} \qquad (2.13)$$

Now let us $\boxed{\text{choose}}$ an initial toric Kähler potential $\boxed{K_0} \in C^\infty(\mathbb{C}^{*n})$

$$K_0(w) = K_0(e^{y+i\theta}) = u_0(y) \qquad (2.14)$$

K_0 defines a toric symplectic, Kähler form $\omega = \frac{i}{2\pi}\partial\bar\partial K_0$ on X_P, a moment map generating the Hamiltonian $\mathbb{T}^n$ action on X_P and a Legendre transform L_{u_0} related by

$$\begin{tikzcd}
(\mathbb{C}^*)^n \arrow[d, "\mathrm{Log}"'] \arrow[dr, "\check{\mu}_0"] & \\
i\,\mathrm{Lie}(\mathbb{T}^n) \cong \mathbb{R}^n \arrow[r, "L_{u_0}"'] & \check{P} \subset \mathbb{R}^n \cong (\mathrm{Lie}\,\mathbb{T}^n)^*
\end{tikzcd} \tag{2.15}$$

$$\check{P} \ni x = \mu_0(w) = \frac{\partial u_0}{\partial y}(y)$$

complex coordinates

symplectic coordinates

$\boxed{\text{Ex. 2.2}}$ Let $M = \mathbb{CP}^2$ and choose as K_0 the Fubini–Study Kähler potential

$$\begin{aligned} K_0(w) &= \frac{1}{2}\log\left(1 + |w_1|^2 + |w_2|^2\right) \\ &= \frac{1}{2}\log\left(1 + e^{2y_1} + e^{2y_2}\right) = u_0(y) \end{aligned} \tag{2.16}$$

Then the symplectic or action coordinates are the components of the moment map

$$\begin{cases} x_1 = \dfrac{\partial u_0}{\partial y_1} = \dfrac{e^{2y_1}}{1+e^{2y_1}+e^{2y_2}} = \dfrac{|w_1|^2}{1+|w_1|^2+|w_2|^2} \\[2ex] x_2 = \dfrac{\partial u_0}{\partial y_2} = \dfrac{e^{2y_2}}{1+e^{2y_1}+e^{2y_2}} = \dfrac{|w_2|^2}{1+|w_1|^2+|w_2|^2} \end{cases} \tag{2.17}$$

with the expected inequalities

$$x_1, x_2 \geq 0 \quad \text{and} \quad x_1 + x_2 \leq 1 \qquad (2.18)$$

Now, to relate the $\mathrm{Log}_t = \frac{\mathrm{Log}}{s}$ map with μ_{is}, we have two (families of) options

O1 [BFMN11] Choose a smooth (on P) convex function of the x's h (eg $h = \|\mu\|^2 = x_1^2 + \cdots + x_n^2$) and find the corresponding geodesics: $\boxed{K_0,\ K_{is}' = h}$

$$h \rightsquigarrow K_{is} \rightsquigarrow \mu_{is} \qquad (2.19)$$

Relate μ_{is} with the $\frac{\mathrm{Log}}{s}$ map.

O2 (Florentino-M-Nunes, work in progress)

Choose as h a symplectic potential which is a convex but only C^0 function of the x's. Then repeat the steps of O1

$$\underbrace{h}_{C^0} \rightsquigarrow \underbrace{K_{is}}_{\text{Kähler metrics with cone angle singularities}} \rightsquigarrow \underbrace{\mu_{is}}_{C^0} \qquad (2.20)$$

We obtain the following results

Thm 2.3 [BFMN11] Let $h = \frac{\|\mu\|^2}{2} = \frac{1}{2}(x_1^2 + \cdots + x_n^2)$

Then
$$\boxed{A_{lim} = \pi(A_{TROP})}$$

where
$$A_{TROP} = \lim_{t\to\infty} \operatorname{Log}_t(Z) \qquad (2.21)$$

and
$$A_{lim} = \lim_{s\to\infty} \mu_{is}(Z)$$

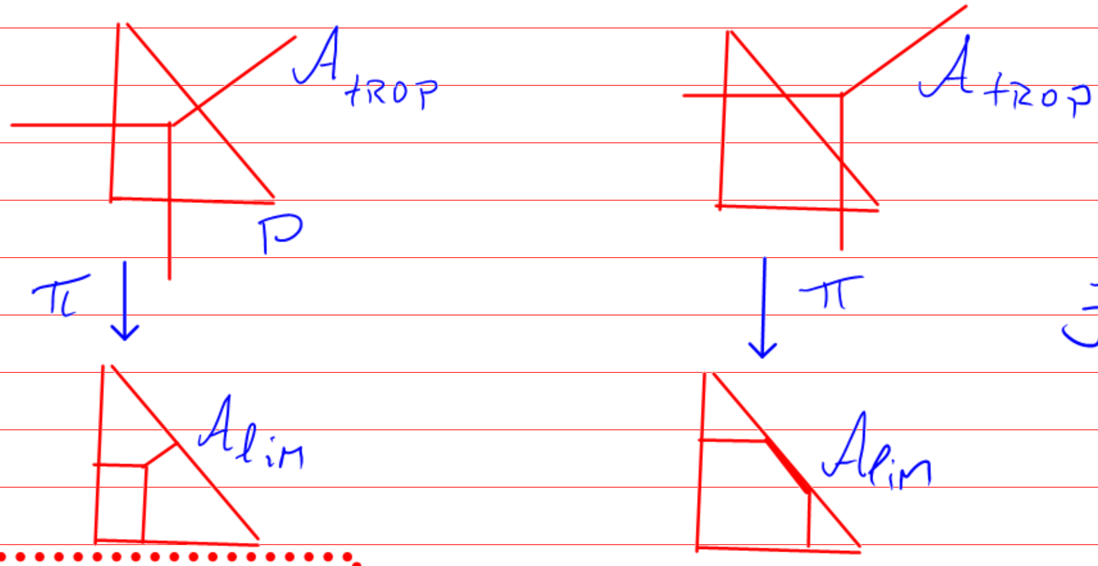

fig. 2.1

$\square$

Thm 2.4 [Florentino - M - Nunes, 2015] Let $h = g_0(x)$ (the symplectic potential, ie the Legendre transform of the Kähler potential u_0. Then on $(\mathbb{C}^*)^n$ μ_{is} and Log_t differ by the s-independent map L_{u_0}. This extends to all faces

$$\begin{tikzcd} & \mathbb{C}^{*n} \arrow[dl, "\mu_{is}"'] \arrow[dr, "\operatorname{Log}_t"] & \\ P & & \mathbb{R}^n \arrow[ll, "L_{u_0}"] \end{tikzcd} \qquad \begin{tikzcd} & \mathbb{C}^{*n_F} \arrow[dl, "\mu_{is}"'] \arrow[dr, "\operatorname{Log}_t"] & \\ F & & \mathbb{R}^{n_F} \arrow[ll, "L_{u_0}"] \end{tikzcd} \qquad (2.22)$$

$\log t = s + i$

$\square$

3 Imaginary time in the quantum theory.

App. 3: Studying the unitary equivalence of different quantizations and coherent state transforms.

[Sn80, ADW91, GS91, LS91, Th96, Wo97, Ha02, B11, HK11, GS11, GS12, LS12, KMN13, MN15, Ze15]

Returning to the simplest symplectic Manifold

$$(M,\omega) = (\mathbb{R}^2, dx \wedge dp) \quad (3.1)$$

and its simplest quantization

$$\psi_{Sch} \in L^2(\mathbb{R}, dx) = \mathcal{H}_{Sch} \quad (3.2)$$

the Schrödinger quantization

$$\hat{x}^{Sch} = x$$
$$\hat{p}^{Sch} = i\hbar \frac{d}{dx} \quad (3.3)$$

To treat the Harmonic oscillator i.e. the Hamiltonian

$$H_{ho} = \frac{1}{2}(p^2 + x^2) \quad (3.4)$$

and in fact for applications all over Quantum Field Theory it is usefull another representation the Fock-Segal-Bargmann representation with a Hilbert space of holomorphic L^2 functions

$$\Psi_F \in \mathcal{H}L^2(\mathbb{R}^2, \nu\, dx\, dp) = \mathcal{H}_F \tag{3.5}$$

$$\nu = e^{-p^2} \tag{3.6}$$

Segal and Bargmann introduced in the begining of the 60's a unitary transform

$$\Psi_{Sch}(x) \xmapsto{U} \Psi_F(x + ip) \tag{3.7}$$

called coherentstate or Fock-Segal-Bargmann transform

$$\boxed{\delta(x - x_0)} \xmapsto[e^{\frac{\Delta}{2}}]{\text{Quantum evol.}} \boxed{\frac{1}{\pi} e^{-\frac{(x-x_0)^2}{2}}} \xmapsto[\substack{\text{analytic} \\ \text{cont.} \\ = \\ \mathcal{C}}]{\substack{\text{classical} \\ \text{evolution} \\ =}} \boxed{\frac{1}{\pi} e^{-\frac{(x+ip-x_0)^2}{2}}} \tag{3.8}$$

$$\boxed{U = \mathcal{C} \circ e^{\frac{\Delta}{2}}} \tag{3.9}$$

Notice that these two steps are in fact intimately related:

$$(\mathcal{C} f)(x) = f(x + ip) = e^{ip\frac{\partial}{\partial x}} f(x) \tag{3.10}$$

We see that

$$e^{\tau X_H} f(x)\Big|_{\tau = i} = e^{i X_H} f(x),$$

where $H = \frac{p^2}{2}$, $X_H = p\,\partial/\partial x$. On the other hand since $-\frac{\Delta}{2} = \frac{(\hat{p}^{Sch})^2}{2} = \hat{H}^{Sch}$

$$e^{-i\tau \hat{H}^{Sch}}\Big|_{\tau = -i} = e^{\Delta/2} \tag{3.11}$$

We see in (3.9), (3.10), (3.11) that the Segal-Bargmann coherent state transform establishing the unitary equivalence of the Schrödinger representation with the Fock representation factorizes into a product of a quantized complex symplectomorphism at time $-i$ followed by classical complex symplectomorphism at time $+i$.

These values of complex time are of course not a coincidence and have a representation theoretic meaning to which we will return in the minicourse.

Q Can the complex time evolution in (3.10) (corresponding to analytic continuation from $\mathbb{R}$ to $\mathbb{C}$) be cast in the framework of Fig. 1.1 and the example of previous section?

In fact the analytic continuation in (3.10) corresponds, quite naturally to the 'inverse' of the mid-point collapse at $\tau = \frac{\pi}{2} i$ in Fig 1.1

$$\tau = -i \rightsquigarrow \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

as that collapse corresponded

to restricting holomorphic functions in $\mathbb{C}$ to functions on $\mathbb{R} \hookrightarrow \mathbb{C}$.

So, one (important) instance of appearance of complex time evolution in quantum physics is to study the unitary equivalence of different quantizations in which different observables act as multiplication operators

$$\mathcal{H}_{Sch} \quad : \quad \hat{x}^{Sch} = x \qquad (3.12)$$

$$\downarrow U = \mathcal{C} \circ e^{\Delta/2}$$

$$\mathcal{H}_F \quad : \quad \hat{z}^F = \widehat{x+ip}^F = z = x+ip$$

The example described above generalizes as in geometric quantization one quantizes a (integral) symplectic manifold by chosing a maximal sheaf of complex valued Poisson commuting functions $\mathcal{O}_P$ (called a polarization) and making them act "diagonally"

$$\rho_{Sch} : \quad (\hat{x}^{Sch} \psi_{Sch})(x) = x \, \psi_{Sch}(x) \qquad (3.13)$$

$$\rho_F : \quad (\widehat{x+ip}^F \psi_F)(z) = z \, \psi_F(z)$$

Then complex symplectomorphisms are used to act on polarizations

$$P_1 \longrightarrow P_2 \qquad (3.14)$$

in an attempt to construct unitary maps establishing the unitary equivalence of different quantizations

$$U_{21} : \mathcal{H}_{P_1} \longrightarrow \mathcal{H}_{P_2} \qquad (3.15)$$

like in the Segal-Bargmann transform in (3.9), (3.12) above.

Thiemann in [Th96] found a complex symplectomorphism mapping the $SU(2)$ spin connection to the $SL(2,\mathbb{C})$ Ashtekar connection in order to study the unitary equivalence of the representations on which these connections act diagonally.

Graefe, Schubert and collaborators bring „complex symplectomorphisms to "life" by showing that in non isolated systems with effective non Hermitian quantum Hamiltonians the evolution of the expectation values of $(\hat{x}, \hat{p})$ for coherent states and also the evolution of the intrinsic geometry of these states satisfy, in first order in $\hbar$ the system (1.8) [GS11, GS12],

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