Lectures 4 and 5

LECTURE 4

Main Refs used in Lecture 4

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[HK] B. Hall, W. Kirwin, Math. Ann 350 (2011) 455

[KMN] W. Kirwin, J. Mourão, J. Nunes, J. Fun. Anal (2013) 1460; J. Math. Phys. 55 (2014) 102101

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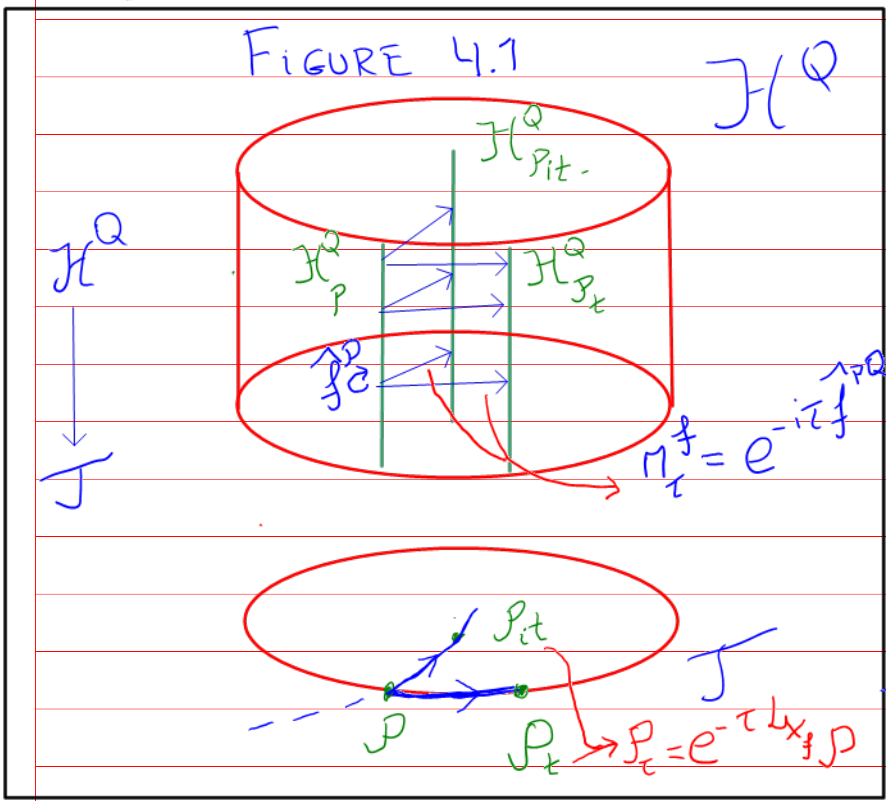
Lecture 4, 19/11/2014

Complex Symplectomorphisms and (generalized) Coherent State Transforms

4.1 Introduction



A quantization of (M,ω) $\Longleftrightarrow$ Choice of

$$P \overset{loc}{=} \langle X_{F_1}, \dots, X_{F_n} \rangle \quad (4.1)$$

$$\{F_i, F_j\} = 0$$

Hilbert Space

$$\mathcal{H}^Q_P = \left\{ \psi \in \mathcal{H}^{PQ} : \nabla_{X_{F_j}} \psi = 0, \forall j \right\} = \left\{ \Psi(F_1, \dots, F_n) e^{iG} \right\} \quad (4.2)$$

If θ is real and part of the F's are real and the other part is complex (constant rk in a dense subset) then

$\boxed{2i\, \mathrm{Im}\, G = K_P}$ = Kähler Potential for P in the Kähler directions does not depend on the real F's

$$\boxed{\mathrm{Re}(G) = \text{gauge choice}} \quad (4.3)$$

Observables

$$\hat{f}^{PQ} = -i\hbar \nabla_{X_f} + f = -i\hbar X_f - \underbrace{(\theta(X_f) - f)}_{L_f} \quad (4.4)$$

Which of these act on $\mathcal{H}^Q_{\langle X_{F_1},\dots,X_{F_n}\rangle}$?

$h(F_1,\dots,F_n)$ (4.5a)

if defined globally on M acts as multiplication operator.

(4.5b)

$\sum_j \alpha_j(F)\,\pi_j \qquad \{F_j,\pi_k\}=\delta_{jk}$

act as first order differential operators.

For all other observables such that

P-modifiers f $\boxed{f:\ L_{X_f}\mathcal{P}\not\subset\mathcal{P}}$ (4.6)

we have two options

Op1 Define $\boxed{\hat{f}^{g}\in\mathcal{H}^Q_P}$ by using functional calculus and factor ordering. E.g.

$$\widehat{\pi_j^2}^{P}=\left(\hat{\pi}_j^{PQ}\Big|_{\mathcal{H}^Q_P}\right)^2 \tag{4.7}$$

$$=\left(-i\hbar\frac{\partial}{\partial F_j}-L_{\pi_j}\right)^2\in\mathcal{H}^Q_P$$

Op2 Use f to change the polarization

$$\mathcal{P}\mapsto\mathcal{P}_t=e^{-tL_{X_f}}\mathcal{P}=$$

$$=\langle X_{\underbrace{e^{-tX_f}F_1}_{=F_1^{-t}}},\dots,X_{\underbrace{e^{-tX_f}F_n}_{=F_n^{-t}}}\rangle \tag{4.8}$$

and then use $\hat{f}^{PQ}$ to lift this action to the Qbundle

$$e^{-i\frac{t}{\hbar}\hat{f}^{PQ}} = e^{\frac{i}{\hbar}\int_0^t L_f(y_{-t'})dt'} e^{-tX_f} \quad (4.9)$$

This indeed lifts the action (4.8) to the quantum bundle

$$M_t^f := e^{-it/\hbar \hat{f}^{PQ}} : \mathcal{H}_P^Q \longrightarrow \mathcal{H}^Q_{P_t} \quad (4.10)$$

$$\Psi(F_1, \dots, F_n) e^{iG} \longmapsto \Psi(F_1^{-t}, \dots, F_n^{-t}) e^{iG_t}$$

where

$$G_t = e^{tX_f}(G) + \frac{1}{\hbar}\int_0^t L_f(y_{\tilde{t}})d\tilde{t}$$

If f and t are real then P and P_t are of the same type and the map M_t^f in (4.10) is a unitary isomorphim mapping between quantization spaces for isomorphic polarizations of the same type in (4.8).

$\boxed{\text{More Interesting}}$

However if f and/or t are complex then (4.8), (4.11) may still make sense.

These are the cases of complex symplectomorphisms and their generalizations.

We will be particularly interested in this lecture, in the case when P is areal or mixed polarization and P_{it}

$$P = \langle X_{F_1}, \dots, X_{F_n} \rangle \mapsto P_{it} = e^{-itL_{X_f}} \langle X_{F_1}, \dots, X_{F_n} \rangle$$

$$= \langle X_{F_1^{-it}}, \dots, X_{F_n^{-it}} \rangle \quad (4.11)$$

where $F_j^{-it} = e^{-itX_f} F_j$ is Kähler for all $t>0^d$. Then

$$\mathcal{H}^Q_{\mathcal{P}_{it}} = \left\{ \underbrace{\psi(F_1^{-it}, \dots, F_n^{-it})}_{\text{holomorphic}} e^{-\frac{K_{it}}{2}} \right\} \qquad (4.12)$$

$Re(K_{it})$ is the Kähler potential

so that f is a $\mathcal{P}$-complexifier as in [1].

We are interested in comparing $\mathcal{H}^Q_{\mathcal{P}}$ and $\mathcal{H}^Q_{\mathcal{P}_{it}}$

As we will see this will lead to a geometric quantization explanation of the unitarity of the Hall coherent state transform and to natural generalizations.

f hits twice

Let us suppose that a $\hat{f}^{\mathcal{P}}$ has been chosen eventhough f does not preserve $\mathcal{P}$.

$$\begin{array}{ccc} f & \longmapsto & \hat{f}^{\mathcal{P}} \circlearrowright \mathcal{H}^Q_{\mathcal{P}} \\ \downarrow & & \\ \hat{f}^{\mathcal{P}Q} \circlearrowright \mathcal{H}^{\mathcal{P}Q} & & \end{array} \qquad (4.13)$$

$\boxed{\text{will need both for CST!}}$

and let us also assume that we have a $*$-Poisson algebra $\mathcal{A}$ acting irreducibly on $\mathcal{H}^Q_{\mathcal{P}}$.

$$
\begin{array}{ccc}
\mathcal{H}^{Q}_{P} & \xrightarrow{\;M^{f}_{t}=e^{-it\hat{f}^{PQ}}\;} & \mathcal{H}^{Q}_{P_t} \\
\circlearrowright & & \circlearrowright \\
\mathcal{A} & & \mathcal{A}_{-t}=e^{-tX_f}\mathcal{A}
\end{array}
\tag{4.14}
$$

Though for real f, M^{f}_{t} is a unitary isomorphism $\boxed{\text{it is not}}$ a isomorphism establishing equivalence of the quantizations P and P_t because it is not intertwining Reps of the same algebra.

GENERAL STRATEGY TO RELATE QUANTIZATIONS

In order to get a representation of $\mathcal{A}$ also on $\mathcal{H}^{Q}_{P_t}$ let us use $\hat{f}^{P}$ first to obtain a representation of $\mathcal{A}_t = e^{tX_f}\mathcal{A}$ on $\mathcal{H}^{Q}_{P}$ using P-evolution and then evolve with M^{f}_{t}:

$$
\begin{tikzcd}
\mathcal{H}^{Q}_{P} \arrow[r, "\mathcal{E}^{f}_{t}=e^{-it\hat{f}^{P}}"] \arrow[rr, bend left, "U^{f}_{t}"] & \mathcal{H}^{Q}_{P} \arrow[r, "M^{f}_{t}=e^{+it\hat{f}^{PQ}}"] & \mathcal{H}^{Q}_{P_t}
\end{tikzcd}
\tag{4.15}
$$

$$
\begin{array}{ccc}
\circlearrowright & \circlearrowright & \circlearrowright \\
\mathcal{A} & \mathcal{A}_t & \mathcal{A} \\
 & \parallel & \\
 & e^{tX_f}\mathcal{A} &
\end{array}
$$

The operator intertwining representations of $\mathcal{A}$ on $\mathcal{H}^{Q}_{P}$ and $\mathcal{H}^{Q}_{P_t}$ is then

$$
\begin{array}{ccc}
U^{f}_{t} = M^{f}_{t}\circ\mathcal{E}^{f}_{t} : \mathcal{H}^{Q}_{P} & \longrightarrow & \mathcal{H}^{Q}_{P_t} \\
\circlearrowright & & \circlearrowright \\
\mathcal{A} & & \mathcal{A}
\end{array}
\tag{4.16}
$$

To "double" the directions of quantizations we can reach from a given one we need to consider $t \in \mathbb{C}$ and in particular $t \rightsquigarrow it \in i\mathbb{R}$ (corresponding to geodesics in the space of Kähler polarizations. The diagram (4.15) and (4.16) become

$$\begin{tikzcd} \mathcal{H}^Q_P \arrow[r, "\mathcal{E}^f_{it} = e^{-t\hat{f}^P}"] \arrow[rr, bend left, "U^f_{it}"] & \mathcal{H}^Q_P \arrow[r, "M^f_{it} = e^{+t\hat{f}^{PQ}}"] & \mathcal{H}^Q_{P_{it}} \end{tikzcd} \qquad (4.17)$$

$$\circlearrowright \mathcal{A} \qquad \circlearrowright \mathcal{A}_{it} = e^{itX_f}\mathcal{A} \qquad \circlearrowright \mathcal{A}$$

The operator intertwining (irr) representations of $\mathcal{A}$ on $\mathcal{H}^Q_P$ and $\mathcal{H}^Q_{P_{it}}$ is than

$$U^f_{it} = M^f_{it} \circ \mathcal{E}^f_{it} : \mathcal{H}^Q_P \longrightarrow \mathcal{H}^Q_{P_{it}} \qquad (4.18)$$

$$\circlearrowright \mathcal{A} \qquad\qquad \circlearrowright \mathcal{A}$$

This then seems to imply* that U^f_{it} is projectively unitary eventhough its factors are highly non unitary. (??)

In fact as we will see in examples projective unitarity holds in general only asymptotically in $\hbar \to 0$. It holds exactly for special f's on $\mathbb{R}^{2n}$ and T^*K.

$\boxed{\text{What fails?}}$

The star relations of the algebra $\mathcal{A}$ may not be preserved by U^f_{it} (only asymptotically as $\hbar \to 0$)

4.2 Examples

Summarizing in the examples we will fix (M,ω) and

(i) Choose P, f find $\mathcal{H}^Q_P$

$$P = \langle X_{F_1}, \dots, X_{F_n} \rangle$$

$$f : \quad L_{X_f} P \not\subset P$$

$$\mathcal{H}^Q_P = \{ \psi(F_1, \dots, F_n)\, e^{iG} \} \quad (4.19)$$

(ii) Find P^f_τ and $\mathcal{H}^Q_{P^f_\tau}$

$$P^f_\tau = e^{-\tau L_{X_f}} P = \langle X_{F_1^{-\tau}}, \dots, X_{F_n^{-\tau}} \rangle \quad (4.20)$$

$$F^\tau = e^{\tau X_f} F$$

$$\mathcal{H}^Q_{P^f_\tau} = \{ \psi(F_1^{-\tau}, \dots, F_n^{-\tau})\, e^{iG_\tau} \} \quad (4.21)$$

(iii) Choose $\hat{f}^P$ and find $\hat{f}^{PQ}$ and U^f_τ

$$U^f_\tau = e^{-i\tau \hat{f}^{PQ}} \circ e^{i\tau \hat{f}^P} : \mathcal{H}^Q_P \to \mathcal{H}^Q_{P^f_\tau} \quad (4.22)$$

(iv) Check Unitarity

Generalized Coherent State transform OR Kostant - Souriau - Heisenberg (KSH) map.

Example 1

$$M = \mathbb{R}^2 \ni (q, p) \qquad \theta = p\,dq$$

(i) $\boxed{\mathcal{P},\ f,\ \mathcal{H}^Q_{\mathcal{P}}}$

$\dashrightarrow$ $\mathcal{P} = \mathcal{P}_{sch} = \langle X_q \rangle = \langle \frac{\partial}{\partial p} \rangle$

$\dashrightarrow$ $f = f(p)$ strictly convex, $f''(p) > 0$

$\dashrightarrow$ $$\mathcal{H}^Q_{\mathcal{P}_{sch}} = \{\psi(q)\} \simeq L^2(\mathbb{R}, dq) \qquad (4.23)$$

(ii) $\boxed{\mathcal{P}^f_\tau,\ \mathcal{H}^Q_{\mathcal{P}^f_\tau}}$

$X_f = f'(p)\frac{\partial}{\partial q}$; $q^{-it} = e^{-itf'(p)\frac{\partial}{\partial q}}(q) = q - itf'(p)$

$\dashrightarrow$ $\mathcal{P}^f_{it} = \langle X_{z_{-it}} \rangle$, $z_{-it} = q - it\,f'(p)$

$\dashrightarrow$ (4.10) $$\mathcal{H}^Q_{\mathcal{P}^f_{it}} = \left\{ \psi(z_{-it})\, e^{-t(pf'(p) - f(p))} \right\} \qquad (4.24)$$

$$= \mathcal{H}L^2_{\mathcal{P}_{it}}(\mathbb{R}^2, dV_t)$$

$K_{it} = 2tL_f(p)$

where $dV_t = e^{-2t\,L_f(p)}\, dq \wedge dp$

(iii) $\boxed{\hat{f}^{\mathcal{P}_{sch}},\ \hat{f}^{pQ},\ U^f_{it}}$

$\dashrightarrow$ $$\hat{f}^{\mathcal{P}_{sch}} = f\left(-i\frac{\partial}{\partial q}\right) \circlearrowleft \mathcal{H}^Q_{Sch} = L^2(\mathbb{R}, dq)$$

$$\hat{f}^{pQ} = -i\,f'(p)\frac{\partial}{\partial q} - L_f(p) \qquad (4.25)$$

$$U^f_{it} = M^f_{it} \circ \mathcal{E}^f_{it} = e^{t\hat{f}^{pQ}} \circ e^{-t\hat{f}^{\mathcal{P}_{sch}}} \iff$$

$\dashrightarrow$ $$U^f_{it} = e^{-tL_f(p)}\, \underbrace{e^{-itf'(p)\frac{\partial}{\partial q}}}_{\text{go to positive Kähler}}\, \underbrace{e^{-t\,f(-i\frac{\partial}{\partial q})}}_{L^2 \longrightarrow \text{Real analytic}} \qquad (4.26)$$

(iv) $U^f_{it}\ :\ \psi(q) \longmapsto U^f_{it}(\psi)(q,p) =$

$$= \left(e^{-tf(-i\frac{\partial}{\partial q})}\psi\right)(q - itf'(p))\, e^{-tL_f(p)} \qquad (4.27)$$

Unitarity of Hall CST for $T^*\mathbb{R}$

Thm For $f = \frac{a}{2}p^2$, $t \geq 0$ U^f_{it} is unitary up to a factor. It is unitary if we take into account the half-form (or metaplectic) correction (see next example). For f non quadratic it is semiclassically unitary but not exactly unitary

$\boxed{\text{Rmk No need to add "averaged heat kernel measure on } \mathbb{C}\text{"}}$

Example 2

$M = T^*K$ K ~ compact Lie group

$$T^*K \cong K \times \mathfrak{k} \ni (x, y) \qquad (4.28)$$

$$\Theta = \sum_j p_j dq_j = \sum_j y_j \underbrace{\omega_j}_{\text{left invariant forms on } K \text{ pulled back to } T^*K}$$

$$\omega = -d\Theta$$

(i) $\boxed{\mathcal{P},\ f,\ \mathcal{H}^Q_{\mathcal{P}}}$

--> $\mathcal{P} = \left\langle X_{F(x)},\ F \in C^\infty(K)\right\rangle = \left\langle \frac{\partial}{\partial y_j}, j=1,..,n\right\rangle$

$$= \mathcal{P}_{Sch} \qquad (4.29)$$

--> $f(Y)$ – Ad-invariant and strictly convex (thus function of Casimirs)

$$X_f = \sum_j \frac{\partial f}{\partial y_j} \underbrace{X_j}_{\text{left inv. vector fields on } K:\ \omega_j(X_k) = \delta_{jk}} \qquad (4.30)$$

$$\dashrightarrow \quad \mathcal{H}^Q_{\mathcal{P}_{Sch}} = \Big\{ \psi(x)\,\underbrace{\sqrt{dx}}_{\text{half form correction}} \Big\} \cong L^2(K, dx) \qquad (4.31)$$

(ii) $\boxed{\mathcal{P}^f_{it},\ \mathcal{H}^Q_{\mathcal{P}^f_{it}}}$

$$\mathcal{P}^f_{it} = e^{-it\,\mathrm{L}_{X_f}}\,\mathcal{P}_{Sch} = \Big\{ X_{e^{-itX_f}F},\ F \in C^\omega(K) \Big\} \qquad (4.32)$$

Using (4.30) and e.g. Peter Weyl thm we obtain, for F real analytic

$$e^{-itX_f} : F(x) \longmapsto F\left(x\, e^{-it\frac{\partial f}{\partial y}}\right) \underbrace{= F(z_{-it})}_{\substack{\text{analytic continuation of } F \\ \text{from } K \text{ to } T^*K}} \qquad (4.33)$$

$$\boxed{z_{-it} = x\, e^{-it\frac{\partial f}{\partial y}}}$$

These functions define a unique complex structure J_{it} on T^*K for which they are holomorphic

$$\dashrightarrow \quad \mathcal{P}^f_{it} = \Big\{ X_{F(z_{-it})},\ F \in C^\omega(K) \Big\} \qquad (4.34)$$

$$\dashrightarrow \quad \mathcal{H}^Q_{\mathcal{P}^f_{it}} = \Big\{ \psi(z_{-it})\, e^{-tL_f} \sqrt{\Omega_{-it}} \Big\} \qquad (4.35a)$$

where

$$\Omega_{-it} = \underbrace{e^{-it\,\mathrm{L}_{X_f}}\ \omega_1 \wedge \dots \wedge \omega_n}_{\substack{\text{top } J_{-it} \text{ holomorphic} \\ K_{\mathbb{C}}\text{-invariant } n \text{ form}}} \qquad (4.35b)$$

$$\|\psi\|^2 = \int_{T^*K} |\psi(z_{-it})|^2\, e^{-2tL_f} \left(\frac{\Omega_{-it} \wedge \bar{\Omega}_{-it}}{(2i)^n\, \omega^n/n!} \right)^{1/2} \frac{\omega^n}{(2\pi)^n\, n!} \qquad (4.36)$$

(iii) $\boxed{\hat{f}^{Psch},\ \hat{f}^{pQ},\ U^f_{it}}$

$$\hat{f}^{pQ} = -i\, X_f - L_f + \underbrace{1 \otimes L_{X_f}}_{\text{action on the half form}} \quad (4.37)$$

From the momentum representation studied in [KW] we choose

$$\widehat{f(y)}^{Psch}\, \pi_{jk}(x) = f(-(\lambda_\pi + \rho))\, \pi_{ij}(x) \quad (4.38)$$

where λ_π is the highest weight of π and ρ is the half summ of the positive roots of K.

For quadratic f: (4.39)

$$f(y) = \frac{\|y\|^2}{2}, \quad \widehat{f(y)}^{Psch} = -\frac{\Delta}{2} + \frac{|\rho|^2}{2}$$

$$\left(C_2(\pi) = |\lambda_\pi + \rho|^2 - |\rho|^2 \right)$$

In this case (4.40)

$$\widehat{f(y)}^{Psch} = \widehat{f(y)}^{SD}$$

SD = Schrödinger–Duflo (see [ST])

$\hat{}^{Psch} \equiv \hat{}^{Sch}$

Then

$$U^f_{it} = e^{t\hat{f}^{pQ}} \circ e^{-t\hat{f}^{Sch}} = \quad (4.41)$$

$$= e^{-tL_f}\, e^{-itX_f}\, e^{-it\, 1\otimes L_{X_f}}\, e^{-t\hat{f}^{Sch}} :$$

So (4.42)

$$\psi(x)\sqrt{dx} \xmapsto{U^f_{it}} \left(e^{-t\hat{f}^{Sch}}\psi\right)(z_{-it})\, e^{-tL_f}\sqrt{\Omega_{-it}}$$

$dx = \omega_1 \wedge \dots \wedge \omega_n$

where z_{-it}: $\pi(z_{-it}) = \pi\left(x\, e^{-it\frac{\partial f}{\partial y}}\right) = e^{-itX_f}\pi(x),\ \forall \pi \in \hat{K}$

(see (4.35b) and (4.38))

iv) Unitarity of U^f_{it}

Thm $\boxed{\text{For } f \text{ quadratic}}$, e.g. $f(y) = \frac{a}{2}\|y\|^2$ then the CST in (4.42) is a unitary isomorphism to the space of $\mathcal{P}^f_{-it}$-holomorphic L^2-sections.

[KW] This transform is $\boxed{\text{equivalent}}$ to the Hall CST.

$\boxed{\text{In general}}$: $\mathcal{P}^f_{-it}$ is Kähler for $\forall t > 0$ and U^f_{it} is asymptotically unitary

[KMN] as $\hbar \to 0$ (but not unitary for f non quadratic).

LECTURE 5

Complex symplectomorphisms and PT-symmetric quantum mechanics

Main Refs used in Lecture 5

[B] C. Bender, Rep. Prog. Phys 70 (2007) 947

[BLU] D. Burns, E. Lupercio, A. Uribe, arXiv 1307.0493

[GS1] E-M. Graefe, R. Schubert, PRA 83 (2011)

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[HHL] D. Huber, E. Heller, R. Littlejohn J. Chem. Phys. 89 (1988) 2003

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[R] C. Rüter et al, Nature Phys., 1515 (2010)

5.1 Complex symplectomorphisms and semiclassical evolution

PT-symmetric quantum mechanics is an alternative approach to quantum mechanics in which emphasis is given to (M,ω) and the choice of a PT-symmetric (possibly complex valued hamiltonian) e.g.

$$h = \frac{p^2}{2} + i q^3$$

PT-symmetry implies that $\hat{h}^{Sch}$ may have a phase with real spectrum possibly at the cost of changing the integration contour in the complexification of the configuration space [B].

For the semiclassical approximation one is then interested in the classical hamiltonian dynamics of PT-symmetric complex valued Hamiltonians.

Complex valued Hamiltonians appear also in the description of dissipative systems, or in general, non isolated systems

Let us first describe the general formalism developed in [GS2] for all, not necessarily PT-symmetric complex Hamiltonians.

In the end we will discuss a PT-symmetric optical wave guide and the PT-symmetric Swanson potential [GS1, GKRS]

Will use Heisenberg coherent states to study the semiclassical behaviour

$$\psi_Y^B(q) = \underbrace{\left(\frac{\operatorname{Im} B}{\pi}\right)^{1/4}}_{=c} e^{iP(q-Q)+\frac{1}{2}B(q-Q)^2} \qquad (5.1)$$

$\langle \hat{q} \rangle = Q$

$\langle \hat{p} \rangle = P$

$Y = (Q,P) \in \mathbb{R}^2$, $B \in \mathbb{H}_1$

A very nice invariant form of states independent of the polarization is the Wigner function (see [L, App B]) of a state $|\psi\rangle$

$$\mathbb{R} \ni W_\psi(y) = \int_{\mathbb{R}^{2n}} \frac{dy'}{(2\pi)^n} \langle \psi | T^+(y') | \psi \rangle e^{i\omega(y',y)} \qquad (5.2)$$

$$\int_{\mathbb{R}^n} W_\psi(q,p) \frac{d^n p}{(2\pi)^n} = |\psi(q)|^2 \qquad (5.3)$$

$$\int_{\mathbb{R}^n} W_\psi(q,p) \frac{d^n q}{(2\pi)^n} = |\tilde{\psi}(p)|^2 \qquad (5.4)$$

The Wigner function of the coherent states (5.1) is also a Gaussian with a metric defined by $B = B_1 + iB_2$

$$W_{\psi_Y^B}(y') = \frac{1}{\pi} e^{-(y'-Y)\cdot G(y'-Y)} \qquad (5.5)$$

where

$$G = \begin{pmatrix} 1 & -B_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} B_2 & 0 \\ 0 & B_2^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -B_1 & 1 \end{pmatrix} \qquad (5.6)$$

Let $\Omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $J = -\Omega G$ a complex structure. The resulting geometry is Kähler for all $B \in \mathbb{H}_1$.

Since we will be acting with complex symplectomorphisms let us consider coherent states (5.1) but with complex centers

$$\Psi^B_{Y^{\mathbb{C}}}(q) = c\, e^{iP^{\mathbb{C}}(q-Q^{\mathbb{C}}) + \frac{1}{2}B(q-Q^{\mathbb{C}})^2} \qquad (5.1)$$

It turns out that states with complex centers $Y^{\mathbb{C}}$, $\tilde{Y}^{\mathbb{C}}$ and the same B such that (5.2)

$$Y^{\mathbb{C}} - \tilde{Y}^{\mathbb{C}} \in L_B = \{(q, Bq),\ q \in \mathbb{C}\}$$

are equal [Huber-Heller-Littlejohn,87] In particular

$$Y^{\mathbb{C}} + L_B \cap \mathbb{R}^{2n} = \{P_J(Y^{\mathbb{C}})\}$$

$$P_J(Y^{\mathbb{C}}) = \mathrm{Re}\, Y^{\mathbb{C}} + J\, \mathrm{Im}\, Y^{\mathbb{C}} \qquad (5.3)$$

More precisely:

Thm 1 [GS2]

$$\Psi^B_{Y^{\mathbb{C}}}(q,p) = e^{i\sigma(Y^{\mathbb{C}}, P_J(Y^{\mathbb{C}}))}\, \Psi^B_{P_J(Y^{\mathbb{C}})}(q,p) \qquad (5.4)$$

$$\sigma(Y_1, Y_2) = \frac{1}{2}(P_1 + P_2)(Q_1 - Q_2)$$

and

$$W_{\Psi^B_{Y^{\mathbb{C}}}}(y') = \frac{e^{-2\,\mathrm{Im}\,\sigma(Y^{\mathbb{C}}, P_J(Y^{\mathbb{C}}))}}{\pi^n}\, e^{-(y' - P_J(Y^{\mathbb{C}}))G(y' - P_J(Y^{\mathbb{C}}))} \qquad (5.5)$$

$\square$

This in particular means that if there is a complex symplectomorphism acting on the centers

$$Y = (Q, P) \mapsto \varphi(Y) = (Q^{\mathbb{C}}, P^{\mathbb{C}}) \quad (5.6)$$

leaving the state approximately Gaussian with metric G then the mean values of $(\hat{q}, \hat{p})$ will change by (5.7)

$$(Q, P) \longmapsto (P_J \circ \varphi)(Q, P) = \mathrm{Re}\begin{pmatrix} Q^{\mathbb{C}} \\ P^{\mathbb{C}} \end{pmatrix} + J\,\mathrm{Im}\begin{pmatrix} Q^{\mathbb{C}} \\ P^{\mathbb{C}} \end{pmatrix}$$
$$= \tilde{\varphi}_J(Q, P)$$

where $J = -\Omega G$

PROP. 1 (8.1 of [MN])

The diffeomorphism $\tilde{\varphi}_J$ is exactly the diffeomorphism corresponding to a complex symplectomorphism and an end complex structure (polarization) J described in the previous section.

RMK There is an important difference with lecture 4: The complex structure here is an attribute of the coherent state whyle in lecture 4 was an attribute of the quantum Hilbert Space $\mathcal{H}^Q_{\mathcal{P}}$ (its chosen polarization). In terms of the generalized CST or KSH map (4.22) (f complex here)

$$U^f_t = \underbrace{e^{-it\hat{f}^{\mathcal{P}Q}}}_{M^f_t} \circ \underbrace{e^{it\hat{f}^{\mathcal{P}}}}_{\mathcal{E}^f_t} : \mathcal{H}^Q_{\mathcal{P}} \longrightarrow \mathcal{H}^Q_{\mathcal{P}^f_t} \quad (5.8)$$

we described the (exact) action of $(\varphi_J)^{-1}$ through M^f_t in lecture 4 and we are now describing the semiclassical action of P_J in this lecture. When f is quadratic the semiclassical action is exact and the KSH map is unitary.

Let f be a campler valued quadratic Hamiltonian

$$f(q,p) = \frac{1}{2} y \cdot H \cdot y \tag{5.9}$$

Then $\mathcal{E}^f_t$ evolves the Gaussian coherent state ψ^B_Y into a Gaussian coherent state

$$\mathcal{E}^f_{-t} = e^{-it\hat{f}^{Sch}} : \psi^B_{\underline{Y}} \longmapsto e^{i\alpha(t)} \psi^{B(t)}_{\underline{Y}^{\mathbb{C}}(t)} \tag{5.10}$$

where $Y(t)$ and $B(t)$

$$\begin{cases} \dot{\underline{Y}}^{\mathbb{C}} = X_f \qquad , \underline{Y}^{\mathbb{C}}(t) = (Q^{\mathbb{C}}, \underline{P}^{\mathbb{C}}) \in \mathbb{C}^2 \\ \dot{B}(t) = -H_{qq} - 2BH_{qp} - B^2 H_{pp} \end{cases} \tag{5.11}$$

$$\left\{ \dot{\alpha} = p\dot{q} - f + \frac{i}{4}\left(H_{pp}B - H_{qq}B^{-1}\right) \right.$$

Using theorem 1 we know that

$$\mathcal{E}^f_{-t} = e^{-it\hat{f}^{Sch}} : \psi^B_{\underline{Y}} \mapsto e^{i\alpha(t)} \psi^{B(t)}_{\underline{Y}^{\mathbb{C}}(t)} = e^{i\tilde{\alpha}(t)} \psi^{B(t)}_{\underline{Y}(t)} \tag{5.12}$$

with real center

$$\underline{Y}(t) = P_{J(t)}(Y^{\mathbb{C}}(t))$$

From [BLU] theorem in p.40 of lecture 3 we know that the real center $Y(t) = P_{J(t)}(Y^{\mathbb{C}}(t))$ satisfies the following equations

$$\begin{cases} \dot{Y} = X_{\mathrm{Re}\, f} + \nabla^{G(t)} \mathrm{Im}\, f \\ \dot{G} = \mathrm{Re}\, H \Omega G - G\Omega \,\mathrm{Re}\, H - \mathrm{Im}\, H + G\Omega^T \mathrm{Im}\, H \Omega G \end{cases} \tag{5.13}$$

also:

$$\left\{ \dot{\tilde{\alpha}} = \ldots \right.$$

5.2 Examples [GS1 and GKRS]

Example 1 Anharmonic Oscillator with damping [GS1]

Let $f(q,p) = h - i\Gamma$

$$h = \frac{\omega}{2}(p^2+q^2) + \frac{B}{4}q^4 \qquad (5.14)$$

$$\Gamma = \frac{\gamma}{2}(p^2+q^2)$$

Rmk Eventhough f is not quadratic the equations for the center and the metric remain approximately valid as long as

$$t : \quad \|G(t)\| \ll 1 \qquad (5.15)$$

We see from (5.13), (5.14) that the motion corresponds to the motion of the anharmonic potential with the gradient term acting like a friction term eventhough the metric is itself evolving.

In the next page see from [GS1] the time evolution of the exact Wigner function (left) and the semiclassical (right) $[\omega = 1, \gamma = 0.2, B = 0.5]$

One interesting rmk in [GS1] is that the Ehrenfest time defined by (5.15) increases with γ.

From Graefe and Schubert [GS1]:

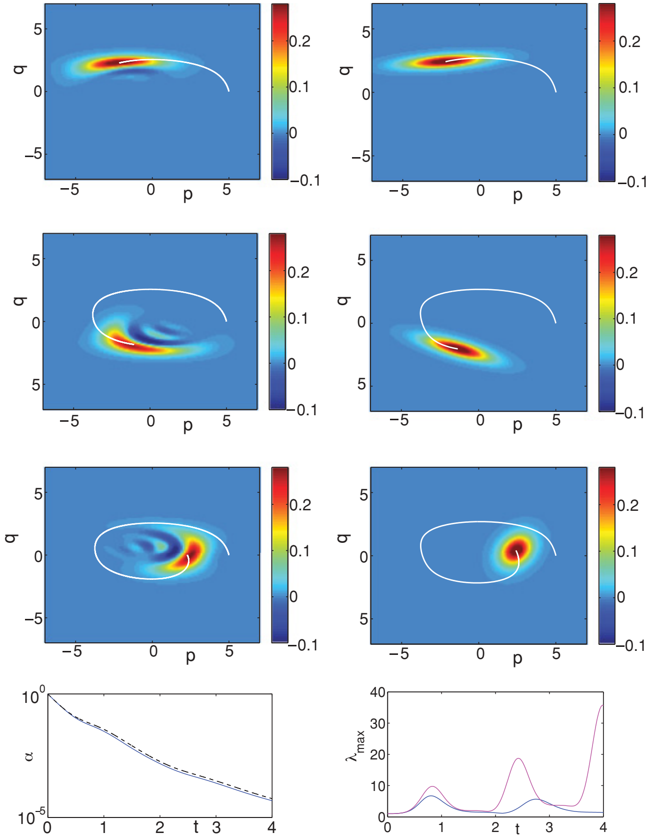


FIG. 1. (Color online) Time evolution of the exact Wigner function (left column) and the semiclassical approximation (right column) for an initial state at $(p,q) = (5,0)$ at different times $(t = 1,2.5,4)$ for the anharmonic oscillator. The white line shows the motion of the center. The left panel on the bottom shows the norm of the exact quantum state (black dashed line) and the semiclassical approximation (blue solid line), and the right panel shows the largest eigenvalue of $G(t)$ (blue line) in comparison with the Hermitian case $\gamma = 0$ (pink upper line).

Example 2 PT symmetric optical wave guide

$$f = h - i\Gamma = \frac{1}{2}(p^2+q^2) - i\tanh(0.2q) \qquad (5.16)$$

This PT-symmetric complex Hamiltonian models a single wave guide with transfer of energy in both directions between the two components.

See figure 2 below (from [CS1]) for the analysis of the evolution and comparison of the exact with the semiclassical.
There is a stable fixed point at $(q,p) = (0,1)$.

Optical model: In the presence of a complex refractive index

$$n(x) = n_R(x) + i\, n_I(x) \qquad (5.17)$$

Satisfying the PT-symmetric conditions

$$n_R(x) = n_R(-x), \quad n_I(x) = -n_I(-x)$$

the electric field envelope satisfies the following paraxial equation of diffraction [Rüter et al, Nature Phys, 2010]

$$i\frac{\partial E}{\partial z} + \frac{1}{2k}\frac{\partial^2 E}{\partial x^2} + k_0[n_R(x) + i n_I(x)]E = 0 \qquad (5.18)$$

$k_0 = 2\pi/\lambda$, $k = k_0 n_0$, n_0 is the substract index in direct analogy with the Schrödinger eq.

FROM [GS1]

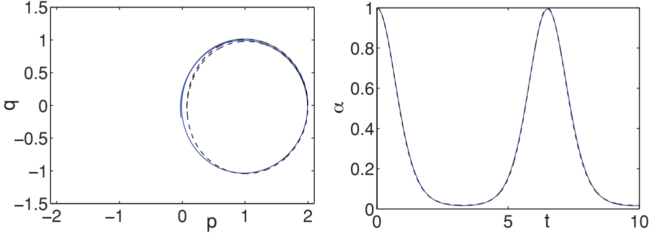


FIG. 2. (Color online) Quantum evolution (black dashed line) versus semiclassical approximation (blue solid line) of a *PT*-symmetric waveguide for an initial state at $(p,q) = (0,2)$. Shown are the phase-space evolution (left) and the evolution of the norm (right).

Example 3 PT-symmetric non-Hermitian Swanson harmonic oscillator [GKRS]

Very interesting is also the complex Hamiltonian flow of the quadratic PT-symmetric Hamiltonian (thus semiclassically exact)

$$f = h - i\Gamma = \frac{\omega_0}{2}(p^2+q^2) - i\delta pq \quad (5.19)$$

From works on PT-symmetric QM it is known that the spectrum is real and

$$E_n = \hbar\omega(n+1/2) \quad 5.20$$

where $\omega = \sqrt{\omega_0^2+\delta^2}$

The equations for the real centers

$$\begin{cases} \dot{y} = X_h - \nabla^G \Gamma \\ \dot{G} = \operatorname{Hess} h\, \Omega G - G\Omega \operatorname{Hess} h + \operatorname{Hess}(\Gamma) - \\ \qquad - G\Omega^T \operatorname{Hess}(\Gamma)\, \Omega G \end{cases} \quad 5.21$$

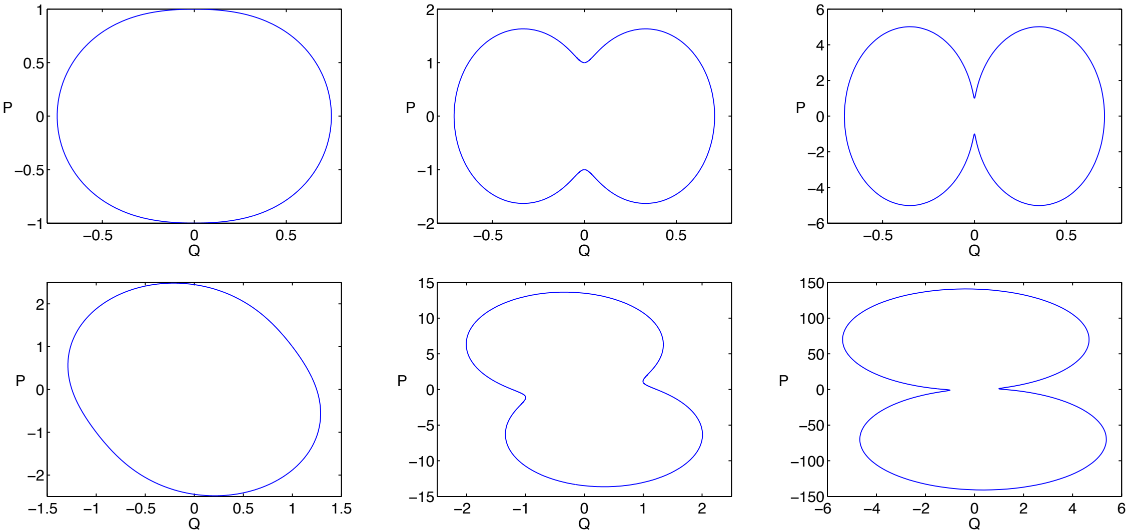
Have the following periodic solutions with period ω [GKRS]

For the metric:

$$G(t) = d(t)\Big[\Big(1+\frac{\delta\omega_0}{\omega^2}(1-\cos(2\omega t))\Big)dq^2 + \Big(1-\frac{\delta\omega_0}{\omega^2}(1-\cos(2\omega t))\Big)dp^2 + \frac{2\delta}{\omega}\sin\omega t\, dq\, dp\Big]$$

with

$$d(t) = \Big(1-\frac{\delta^2}{\omega^2}(1-\cos(2\omega t))\Big)^{-1}$$



FROM [GKRS]

Figure 1. Phase-space trajectories for different parameter values and initial conditions. The parameters are $\omega_0 = 1$ $\delta = 0.5, 0.9, 0.99$ (from left to right). The initial conditions for the top panel are $(P_0, Q_0) = (1, 0)$ and for the bottom panel $(P_0, Q_0) = (1, 1)$.

and

$$Q(t) = d(t)\left(Q_0 \cos(\omega t) - \frac{P_0}{\omega}(\omega_0 - \delta)\sin(\omega t)\right)$$

$$P(t) = d(t)\left(\frac{Q_0}{\omega}(\omega_0 + \delta)\sin \omega t + P_0 \cos \omega t\right)$$

Problems if $\frac{\delta^2}{\omega^2} \geqslant \frac{1}{2} \Leftrightarrow \delta^2 \geqslant \omega_0^2$.

See examples in the figure above.

Thank You!