LECTURES

IQG

FAU ERLANGEN-Nürnberg

Fall 2014

Quantum Mechanics

vs

Kähler Geometry

Lectures 1 and 2

How Kähler geometry enters the picture

LECTURE 1

LECTURE 2

LECTURE 3

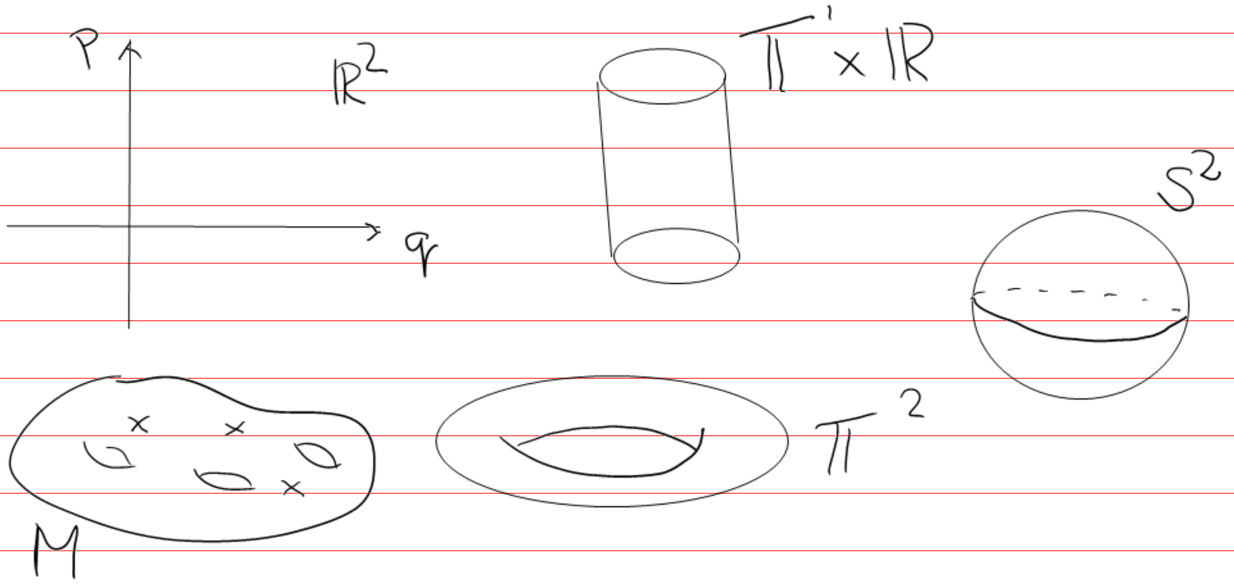
Lecture 1, 29/10/2014

Summary for Lectures 1 and 2

* Classical System — Quantized System

(M, ω) — $(\mathcal{H}^Q, \rho_\hbar)$

* For the summary let us choose $\dim M = 2$



* Choose (locally + gluing) a real observable to "be quantized as multiplication operator"

$$F \in C^\infty(M) \quad (\text{or } C^\infty(U)) \qquad dF_x \neq 0 \text{ on } M \ (\text{or } U)$$

$\sim$ Choose a Real polarization

Then

1) $\mathcal{H}^Q_{\{F\}} \ni \Psi = \Psi(F)\, e^{iG}$ ⇝ common phase function for all wave functions

$\Psi(F) \in L^2(\mathrm{Sp}(\mathcal{A}^Q_F), d\mu)$

2) $\rho_{\hbar}^{\{F\}}(F) = F$

(*) WARNING

Different choices of F may (in general will) lead to INEQUIVALENT quantum theories even in the simplest example of $\mathbb{R}^2$

$$\{F_1 = q\} \;\not\Longleftrightarrow\; \left\{F_2 = \frac{p^2}{2} + V(q)\right\}$$

Sch quant. "Energy Rep quantiz"

For eg V a polynomial of degree higher than two

(*) Mathematically better if instead of choosing a Real observable(s) which we want to quantize as multiplication operator we choose a complex

$$F = F_1 + i F_2 \in C^{\omega}(M, \mathbb{C})$$
(OR $C^{\omega}(U, \mathbb{C})$)

especially if

$$(dF_1 \wedge dF_2)_x \underset{\text{loc}}{\neq} 0 \quad \forall x \in M$$
at every point (OR $\forall x \in U$)

(*) THEN

$F \rightsquigarrow$ Complex structure on J_F compatible with ω

so that

$\gamma = \omega(J\cdot, \cdot)$ is a (kähler) metric with local Kähler Potential K_F

K_F: $\omega = -i \, \partial_{J_F} \bar{\partial}_{J_F} K_F$

2) $\underline{\Psi} = \underbrace{\underline{\Psi}(F)}_{\mathcal{H}_{L_2}} e^{-K_F/\hbar}$ (J_F - holomorphic)

3) $\int_{\hbar}^{F_1+iF_2} (\bar{F}) = F$

1. Dirac's Program $\rightsquigarrow$ Nogo theorems

Some Refs given along

* Classical system: $(\underbrace{\underbrace{M, \omega}_{\text{kinematics}}, H}_{\text{dynamics}})$

* The symplest (already quite rich) examples

$$\left(\mathbb{R}^{2n}, dq \wedge dp = \sum_{j=1}^{n} dq_j \wedge dp_j, H = \frac{1}{2}|p|^2 + V(q)\right) \quad (1.1)$$

* The symplectic form defines an invertible map

$$\mathfrak{X}(M) \underset{\longleftarrow}{\overset{\omega}{\longrightarrow}} \Omega^1(M)$$

$$\omega : X \longmapsto \omega(X, \cdot) \quad (1.2)$$

$$\omega^{-1} : \alpha \longmapsto \omega^{-1}(\alpha)$$

* Then we get a very important map

$$C^\infty(M) \longrightarrow \mathfrak{X}(M) \quad (1.3)$$

$$f \longmapsto X_f := \omega^{-1}(df)$$

$C^\infty(M)$ become a (∞-dimensional) Lie algebra

$$\{f, h\} = -X_f(h) = X_h(f) \quad (1.4)$$

such that the map $f \mapsto X_f$ is a Lie algebra (anti) homomorphism with kernel the constant functions.

* In our example:

$$C^\infty(\mathbb{R}^{2n}) \longrightarrow \mathcal{X}(\mathbb{R}^{2n})$$

$$f \longmapsto X_f = \omega^{-1}(df) = \sum_{j=1}^{n} \frac{\partial f}{\partial p_j}\frac{\partial}{\partial q_j} - \frac{\partial f}{\partial q_j}\frac{\partial}{\partial p_j} \qquad (1.5)$$

and $\{f,g\} = \sum_j \frac{\partial f}{\partial q_j}\frac{\partial g}{\partial p_j} - \frac{\partial f}{\partial p_j}\frac{\partial g}{\partial q_j}$

* Classical Dynamics corresponds to the eqs for the integral curves of X_H.

* In our example

Newton eqs

$$\begin{cases} \dot q_j = \dfrac{\partial H}{\partial p_j} \\ \dot p_j = -\dfrac{\partial H}{\partial q_j} \end{cases} \iff \begin{cases} \dot q_j = p_j \\ \dot p_j = -\dfrac{\partial V}{\partial q_j} \end{cases} \qquad (1.6)$$

* Dirac's dream quantization

$$(M,\omega) \longmapsto (\mathcal{H}, \rho_\hbar)$$

$$C^\infty(M) \ni f \longmapsto \rho_\hbar(f) \circlearrowleft_{s.a.} \mathcal{H} \qquad (1.7)$$

i) $[\rho_\hbar(f), \rho_\hbar(g)] = i\hbar\, \rho_\hbar(\{f,g\})$

ii) $\rho_\hbar(1) = I$, iii) $\rho_\hbar$ is irreducible

* It was soon realized that this dream can never be realized not even for the simplest classical system

Not even for the symplest quantum system

$$(\mathbb{R}^2, dp\wedge dq) \qquad (1.8)$$

[Groenewold, 1946] [van Hove, 1951]
(See [Gotay, JMP 40 (1999)])

2 Geometric Quantization (GQ)

Let us now summarize the way GQ addresses the problem of quantizing a classical system

History: [Souriau, CMP 1 (1966)]
[Kostant, LNM 170 (1970)]

Book: {[Woodhouse, OUP (1997)]
[Tuynman III Erlangen Fall School]

Recent Reviews: [Blau UBern] [Thiemann, CUP 2007, Ch23]
[Nunes, RMathP 2014]

STEP 1:

* 2.1 Prequantization

The first attempt to construct a Dirac Dream quantization "almost" succeeds.

There is a natural Hilbert space associated to a classical system

$$(M,\omega) \quad \mathcal{H}^{PQ} = L^2\left(M, \frac{\omega^n}{(2\pi\hbar)^n n!}\right) \qquad (2.1)$$

and an action of observables as s.a. operators

$$f \longmapsto -i\hbar X_f \circlearrowleft \mathcal{H}^{PQ} \qquad (2.2)$$

Satisfying Dirac's commutation relations (i) but of course not satisfying (ii)

$$1 \longmapsto 0$$

A natural way to try to recover (ii) is by setting

$$f \longmapsto -i\hbar X_f + f \qquad (2.3)$$

We clearly have (ii) but we lost (i). To recover (i) while preserving (ii)

$$f \longmapsto -i\hbar X_f + f - \Theta(X_f) \qquad (2.4)$$

To satisfy (i) one verifies that one has to have

$$d\Theta = \omega \qquad (2.5)$$

This is ok for $\mathbb{R}^{2n}$ and $\omega = \sum_j dp_j \wedge dq_j$ as we can choose

$$\Theta = \sum_j p_j \, dq_j \qquad (2.6)$$

However if M is compact then ω can't be exact and then (2.5) can be true only locally

$$\omega = d\Theta_\alpha \quad \text{on } U_\alpha \qquad (2.7)$$

By gluing these data $(\Theta_\alpha, U_\alpha)$ we can see these 1-forms define

a line bundle $L \longrightarrow M$
with a connection ∇
such that $F_\nabla = \frac{\omega}{2\pi i\hbar} \Rightarrow \left[\frac{\omega}{2\pi\hbar}\right] \in H^2(M,\mathbb{Z})$

$$\nabla \sigma_\alpha = -\frac{i}{\hbar}\, \theta_\alpha\, \sigma_\alpha \qquad (2.8)$$

for a system of trivializing sections $(\sigma_\alpha, U_\alpha)$.

Then the formula (2.4) above defines the following global operator

$$f \longmapsto \hat{f}^{PQ} = -i\hbar \nabla_{X_f} + f \;\circlearrowright\; \Gamma(M,L)$$

$$\hat{f}^{PQ} \underset{loc}{=} -i\hbar X_f - L_f \qquad (2.9)$$

Where the Lagrangian of f is

$$L_f = \Theta(X_f) - f \qquad (2.10)$$

or in $\mathbb{R}^{2n}$

$$L_f = \Big(\sum_j p_j dq_j\Big)\Big(\sum_k \frac{\partial f}{\partial p_k}\frac{\partial}{\partial q_k} - \frac{\partial f}{\partial q_k}\frac{\partial}{\partial p_k}\Big) - f$$

$$= \sum_j p_j \frac{\partial f}{\partial p_j} - f \qquad (2.11)$$

* We choose a ∇-compatible Hermitian structure on L

$$h(\sigma_1, \sigma_2) \in C^\infty(M) \qquad (2.12)$$

$$d\, h(\sigma_1, \sigma_2) = h(\nabla\sigma_1, \sigma_2) + h(\sigma_1, \nabla\sigma_2)$$

This means that if σ_α: $\nabla\sigma_\alpha = -\frac{i}{\hbar}\theta_\alpha\sigma_\alpha$ with θ_α real then

$$d\, h(\sigma_\alpha, \sigma_\alpha) = 0 \Rightarrow h(\sigma_\alpha, \sigma_\alpha) = \text{const}$$

so that on connected domains a connection has a unique compatible Hermitean structure up to a positive factor: the sections σ_α with θ_α real have constant norms $h(\sigma_\alpha, \sigma_\alpha)$. These are the unitary or unit trivializing sections for ∇.

* Then we get a Hilbert space

$$\mathcal{H}^{PQ} = \Gamma_{L^2}(M, L) \qquad (2.13)$$

$$\langle \sigma_1, \sigma_2 \rangle = \int_M h(\sigma_1, \sigma_2) \frac{\omega^n}{(2\pi\hbar)^n}$$

on which the prequantum operators (2.9)

$$\hat{f}^{PQ} = -i\hbar \nabla_{X_f} + f$$
$$= -i\hbar X_f - L_f$$

act as self-adjoint operators

$\mathcal{H}^{PQ}$
"Too Big"

The quantization of observables (2.9) satisfies the Dirac commutation relation (1.7) i) the condition (ii) for constant observables but the representation of $(C^\infty(M), \{,\})$ on $\mathcal{H}^{PQ}$ is not irreducible.

Intuitively also a Hilbert space of L^2 (twisted) functions on the phase space is not expected to respect the uncertainty relations

$$\Delta q\, \Delta p \gtrsim \hbar/2 \qquad (2.15)$$

Also $\dim \mathcal{H}^{PQ} = \infty$ also for compact M while in that case from (2.15) one expects

$$\dim \mathcal{H}^{Q} \sim \frac{\mathrm{vol}_\omega(M)}{\hbar^n} \qquad (2.16)$$

The one parameter group of unitary transformations generated by the prequantization operators (2.9) is very interesting

$$\hat{f}^{PQ} = -i\hbar \nabla_{X_f} + f_{loc} = -i\hbar X_f - L_f$$

$$L_f = \Theta(X_f) - f \underset{M=T^*N}{=} \sum_j p_j \frac{\partial f}{\partial p_j} - f$$

$$\left(e^{-\frac{it}{\hbar}\hat{f}^{PQ}}\psi\right)(x) = e^{\frac{i}{\hbar}\int_0^t L_f^{t'}(x)\, dt'}\, e^{-tX_f}\psi(x)$$

$$= e^{i/\hbar \int_0^t L_f^{t'}(x)\, dt'}\, \psi(x_t) \qquad (2.14)$$

LECTURE 2 , 5/11/2014

2.2 Quantization

* Need to choose $\mathcal{H}^Q \subset \mathcal{H}^{PQ}$ (2.17)

2.2.1 Motivating Example

How? Motivating example

One one hand we have

$$(M,\omega) = (\mathbb{R}^2, dp \wedge dq)$$

$$L = \mathbb{R}^2 \times \mathbb{C} \qquad \mathcal{H}^{PQ} = L^2(\mathbb{R}^2, dq\,dp)$$

$$\Theta = p\,dq \qquad (2.18)$$

$$\hat{f}^{PQ} = -i\hbar X_f - L_f =$$

$$= -i\hbar X_f - \left(p\frac{\partial f}{\partial p} - f\right)$$

$$\hat{p}^{PQ} = -i\hbar \frac{\partial}{\partial q}$$

$$\hat{q}^{PQ} = i\hbar \frac{\partial}{\partial p} + q$$

* On the other we know the Schrödinger representation for this system

$$\mathcal{H}^Q_{Sch} = L^2(\mathbb{R}, dq) \qquad (2.19)$$

$$\hat{q}^{Sch} = q \qquad \hat{p}^{Sch} = -i\hbar \frac{d}{dq}$$

* We see that (2.20)

$$\mathcal{H}^Q_{Sch} = \left\{ \psi \text{ "}\in\text{" } \mathcal{H}^{PQ} : \frac{\partial \psi}{\partial p} = 0 \right\}$$

$$\nabla_{\frac{\partial}{\partial p}} \psi = 0 \Leftrightarrow \nabla_{X_q} \psi = 0$$

Also

$$\hat{q}^{Sch} = \hat{q}^{pQ}\Big|_{\mathcal{H}^Q_{Sch}} = \left(i\hbar\frac{\partial}{\partial p} + q\right)\Big|_{\mathcal{H}^Q_{Sch}} \quad (2.21)$$

$$\hat{p}^{Sch} = \hat{p}^{pQ}\Big|_{\mathcal{H}^Q_{Sch}} = \left(-i\hbar\frac{\partial}{\partial q}\right)\Big|_{\mathcal{H}^Q_{Sch}}$$

* LESSON

The change (2.22)

$$\mathcal{H}^{pQ} = L^2(\mathbb{R}^2, dq\,dp) \rightsquigarrow \mathcal{H}^Q = L^2(\mathbb{R}, dq)$$

corresponds to choosing half of the canonical variables p and take wave functions that do not depend on them

$$\psi(q,p) \underset{\nabla_{X_q}\psi=0}{\overset{\nabla_{\frac{\partial}{\partial p}}\psi=0}{\rightsquigarrow}} \Psi(q) \quad (2.23)$$

This implies that the other half q act as multiplication operators

$$\hat{q}^{Sch} = -i\hbar\nabla_{X_q} + q \quad (2.24)$$

2.2.2 General Case - Real observables

* In general then choose

$$Q_1^{(\alpha)}, \dots, Q_n^{(\alpha)} \text{ on } U_\alpha \quad (2.25)$$

$$\mathcal{H}^Q = \{\psi \in \mathcal{H}^{pQ} : \nabla_{X_{Q_1^{(\alpha)}}}\psi = \dots = \nabla_{X_{Q_n^{(\alpha)}}}\psi = 0\}$$

consistency $\Rightarrow \{Q_i, Q_j\} = 0$

$$\mathcal{H}^{Q} = \{ \Psi(Q_1^{(\alpha)}, \dots, Q_n^{(\alpha)}) \, e^{iF(Q,P)} \}$$

Or more invariantly choose a integrable Lagrangian distribution

$$\boxed{\mathcal{P}} \subset TM \tag{2.26}$$

$\Leftrightarrow$ on U_α $\quad \mathcal{P}|_{U_\alpha} = \langle X_{Q_1^{(\alpha)}}, \dots, X_{Q_n^{(\alpha)}} \rangle$

and define (Real) Polarization

$$\mathcal{H}^{Q}_{\mathcal{P}} = \left\{ \Psi \in \mathcal{H}^{PQ} = \Gamma_{L^2}(M, \mathcal{L}) : \nabla_X \Psi = 0 \;\forall X \in T(\mathcal{P}) \right\} \tag{2.27}$$

2.2.3 Problems and physical interpretation

PROBLEM 1 //

Only the observables f : X_f preserves $\mathcal{P}$ have f^{PQ} preserving $\mathcal{H}^Q_{\mathcal{P}}$ thus defining

$$\hat{f}^{\mathcal{P}} = \hat{f}^{PQ} \Big|_{\mathcal{H}^Q_{\mathcal{P}}} \tag{2.28}$$

Basically

$$f = \sum_j \vartheta_j^{(\alpha)}(Q) \, P_j^{(\alpha)} + b^{(\alpha)}(Q) \tag{2.29}$$

in case they can be glued to a global function.

Physical Interpretation

$\mathcal{H}^Q_{\mathcal{P}}$ — Quantization in which the observables generating the polarization Q_j act diagonally

$$\widehat{f(Q_1,\dots,Q_n)}^{\mathcal{P}} = f(Q_1,\dots,Q_n) \qquad (2.30)$$

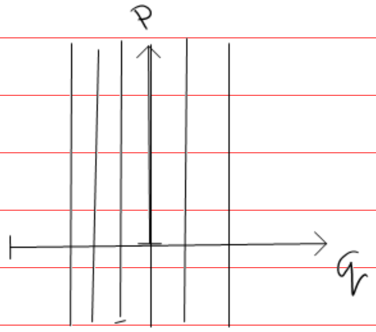
PROBLEM 2

Dependence of quantization on the choice of $\mathcal{P}$

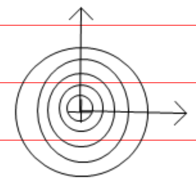
Example 1

$\mathbb{R}^2$

$$\mathcal{P}^{sch} = \langle X_q \rangle = \left\langle \frac{\partial}{\partial p} \right\rangle \qquad (2.31)$$



$$\mathcal{P}^{h.o.} = \langle X_H \rangle = \left\langle \frac{\partial}{\partial \theta} \right\rangle$$



$$H = \frac{1}{2}(p^2 + q^2)$$

$$\mathcal{H}^Q_{\mathcal{P}^{sch}} = L^2(\mathbb{R}, dq)$$

$\downarrow S$ unitarily equivalent $\qquad (2.32)$

$$\mathcal{H}^Q_{\mathcal{P}^{h.o}} = \left\langle \delta\left(H - \hbar(n+\tfrac{1}{2})\right) e^{in\theta} \right\rangle$$

BUT if instead of H^{ho} take something more complicated e.g.

$$H = \frac{p^2}{2} + \frac{q^2}{2} + \frac{\lambda}{4} q^4 \qquad (2.33)$$

then we know that

$$\mathcal{H}^Q_{P_{sch}} \not\simeq \mathcal{H}^Q_{P^H} \qquad (2.34)$$

Non unitarily equivalent $P^H = \langle X_H \rangle$

because $Sp(\hat{H}^{sch}) \neq Sp(\hat{H}^{P^H})$

2.2.4 General case - Complex observables / Polarizations

! ✗ Motivated by Segal - Bargmann representations what if we choose n complex valued functions in involution

$$\{ f_j^{(\alpha)}, f_k^{(\alpha)} \} = 0 \qquad (2.35)$$

as those which we want to act diagonally? They generate a Lagrangian distribution BUT ON the complexified tangent bundle

$$P = \langle X_{f_1^{(\alpha)}}, \dots, X_{f_n^{(\alpha)}} \rangle \subset TM \otimes \mathbb{C} \qquad (2.36)$$

✗ There are two extreme cases

Case 1 Real Polarizations

If

$$P \cap \overline{P} = P \qquad (2.37)$$

Then $\mathcal{P}$ is a "fake" complex polarization or real polarization generated locally by n real valued functions on involution

$$\mathcal{P} = \mathcal{P}_{\mathbb{R}} \otimes \mathbb{C} \qquad (2.38)$$

* Pseudokähler Polarizations

$$\mathcal{P} \cap \overline{\mathcal{P}} = \{0\} \qquad (2.39)$$

$$\Updownarrow$$

$$\mathcal{P} = \langle X_{f_j}, \ j = 1, \ldots, n \rangle$$

$$f_j = u_j + i v_j \ : \ \bigwedge_j du_j \bigwedge_e dv_e \neq 0 \quad \forall x \in M$$

2.2.5 Pseudo Kähler polarizations and pseudo Kähler geometry

Theorem

i) $\exists$ an (integrable) complex structure J on M such that

$$P = P_J = T^{0,1}M = \langle X_{z_1}, \dots, X_{z_n} \rangle \qquad (2.40)$$

where $z_1, \dots, z_n$ are local J-holomorphic coordinates and

ii) (ω, J) form a compatible pair, i.e.

$$\omega(JX, JY) = \omega(X, Y)$$

$\Rightarrow$ $\gamma(X,Y) = \omega(JX, Y)$ is symmetric nondeg.

If $\gamma = \omega(J\cdot, \cdot) > 0 \qquad (2.41)$
then (ω, J) is Kähler.

Proof sketch Define the complex structure as follows

$$(TM_{\mathbb{C}})_x = P_x \oplus \overline{P}_x$$

$$J_x : \quad J_x(u + v) = -iu + iv$$

This defines a almost cx structure which is integrable iff the distributions of eigenspaces $P, \overline{P}$

are integrable which they are by definition.

Compatibility with ω is

$$\omega(\mathcal{J}\cdot,\mathcal{J}\cdot) = \omega$$

From the definition of $\mathcal{P}$ we know that

$$\omega(u_1,u_2) = 0 \quad \text{if} \quad \begin{array}{l} u_1,u_2 \in \mathcal{P} \\ u_1,u_2 \in \overline{\mathcal{P}} \end{array}$$

Then for $u_1 \in \overline{\mathcal{P}}$, $u_2 \in \mathcal{P}$ we have

$$\omega(\mathcal{J}u_1,\mathcal{J}u_2) = \omega(iu_1,-iu_2)$$
$$= \omega(u_1,u_2)$$

$\boxed{\text{Ш}}$

Then $(\omega,\mathcal{J})$ define a pseudo Riemannian metric

$$\gamma(u,v) = \omega(\mathcal{J}u,v)$$

Def A polarization $\mathcal{P}$ is Kähler if

$$\mathcal{P}\cap\overline{\mathcal{P}} = \{0\}$$

and

$$\omega(\mathcal{J}\cdot,\cdot) > 0$$

Its interesting that the deviation from flateness of this kähler metric seems to be measuring the deviation of the real and imaginary parts of locally J-holomorphic functions from being canonically conjugate pairs

$$z_j = f_j + i h_j$$

$$\{ f_j , h_\ell \} = c_{j\ell} = \text{const} \Rightarrow \gamma \text{ is flat}$$

Let now $n=1$ and

$$z = f + ih = f + i F(\pi_f)$$

Then one can show that the scalar curvature is

$$R_\gamma = -\left(\frac{1}{F'}\right)''(\pi_f)$$

$$\Rightarrow \quad = 0$$

$$F = a\pi_f$$

2.2.6 Kähler Quantization

* For the Hilbert space we have

$$\mathcal{H}^Q_{P_J} = \{ s \in \Gamma_{L^2}(M,L) : \nabla_X s = 0 \ \forall X \in \Gamma(T^{0,1}M) \}$$

$$\Gamma(T^{0,1}M) = \Gamma(\bar{D})$$

$$= H^0_J(M,L)$$

= space of $(1,0)$-holomorphic sections

They can be written in the following form

$$\psi = f(z) e^{-K/\hbar} \sqrt{dz}$$

where K is the so called Kähler potential

$$i\partial\bar{\partial} K = i \sum \frac{\partial}{\partial z_j} \frac{\partial}{\partial \bar{z}_l} K \, dz_j \wedge d\bar{z}_l$$

$$= \sum_{j\ell} g_{j\bar{\ell}} \, dz_j \wedge d\bar{z}_\ell$$

$$\gamma = \sum_{j\ell} g_{j\bar{\ell}} \, dz_j \otimes d\bar{z}_\ell$$

2.2.7 Advantages of Kähler Polarizations / Quantizations

(for the quantum theory)

1. Mathematically much better defined

$$\mathcal{H}^{Q}_{P_J} = \{ \psi \in \Gamma_{L^2}(M, L) : \nabla_{\frac{\partial}{\partial \bar{z}_j}} \psi = 0 \}$$

$$\subset \mathcal{H}^{pQ}$$

2. "Functions" on the phase space which gives a better semi-classical understanding.

3. For systems with natural complex observables like the Ashtekar variables (or their Wilson loops) the quantization on the associated polarization is the way to go.

4. Because of being H-space of holomorphic sections the "δ" functions are normalizable

$$K_{z_0} : f \longmapsto f(z_0)$$

OR as one says in Maths these are Hilbert spaces with reproducing kernels.

5. Possibilities which were made possible by Thomas idea of complexifiers:

[Thiemann, CQG 1996]

Since we are dealing with complex observables we betta accept to use complexified canonical transformations

Lecture II

Interestingly a similar need appeared independantly in Kähler geometry

[Semmes, Am J Math 1992]
[Donaldson, Asian J Math (1999)]

5a. Relate quantizations for non equivalent polarizations using a complexified canonical transformation and a CST-like transform

Lecture III

5b. Use complexified canonical transformations to define better quantization in real polarizations.

6. ~~T~~öplitz quantization and its relation with

deformation quantization

$$f \mapsto P_J \circ \hat{f}^{PQ}$$

orthogonal projection on the space of J-holomorphic sections

Lecture 3, 12/11/2014

Space of Quantizations and its Geometry

∞ - dimensional space of (geometric) quantizations of a physical system

$\cong$

Space of kähler structures and their degenerations

3. ∞ dimensional space of quantizations / Polarizations

3.1 Introduction / SUMMARY

* We saw that given a classical system (M,ω) + flat connection there is one prequantization

$$(M,\omega) \rightsquigarrow \left(\mathcal{H}^{PQ} = \Gamma L^2(M,L),\ \rho^{PQ}\right) \quad (3.1)$$

* BUT there is allways an infinite dimensional space of (geometric) quantizations parametrized by the choice of (possibly singular) polarizations

$$\{F_1,\dots,F_n\} : \{F_j, F_k\} = 0 \quad (3.2)$$

$$dF_1 \wedge \dots \wedge dF_n \neq 0 \quad \text{on a dense open subset}$$

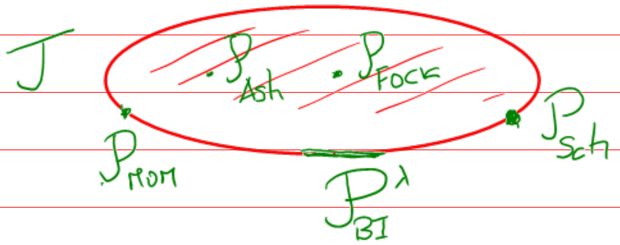
$$\mathcal{P} \underset{loc.}{=} \langle X_{F_1},\dots,X_{F_n} \rangle$$

$\mathcal{P}_{Ash}$ - Ashtekar Polarization

$\mathcal{P}_{BI}^{\lambda}$ - Barbero Immirzi

$\mathcal{P}_{Mom}$ - Momentum Space

$\mathcal{J}$ ~ Space of polarizations ~ space of quantizations



(3.3)

$Int(\mathcal{J})$ ~ Space of Kähler polarizations/quantizations

possibly empty ("usually" not). If not empty then is ∞ dimensional

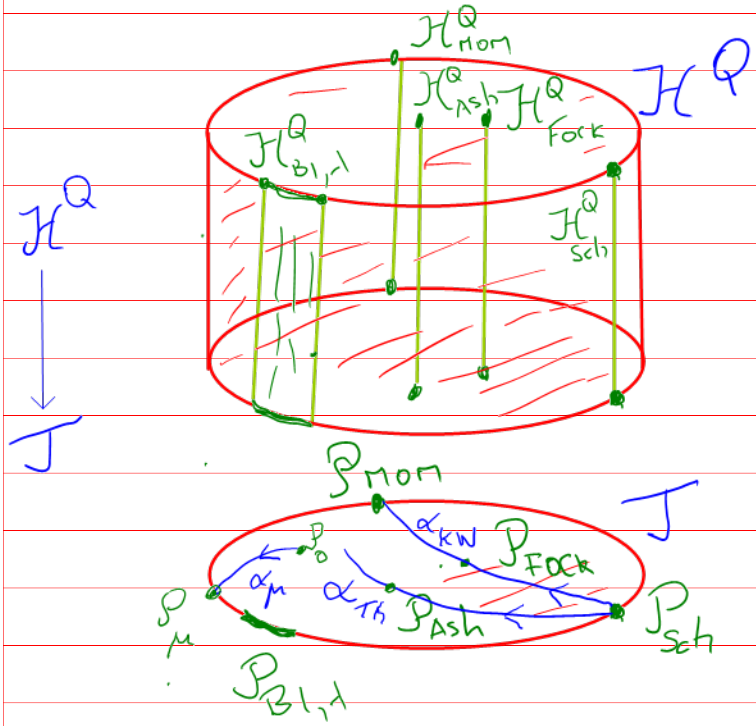
$\partial\mathcal{J}$ ~ Space of (possibly singular) mixed or real polarizations

* So instead of fulfilling the Dirac's dream of having a well defined quantization

$$(M,\omega) \overset{Q}{\rightsquigarrow} (\mathcal{H}, \rho) \quad (3.4)$$

We got ourselves an $\boxed{\text{infinite dimensional}}$ bundle

QUANTUM BUNDLE



(3.5)

α_{Th} - Thiemann geodesic
α_{KW} - Kirwin-Wu geodesic
α_μ - starts at any polarization and ends at $\mathcal{P}_\mu$

* In order to try to make sense out of this apparent mess we consider the following questions

* Q1// Are all these quantum theories equivalent?

A1// No! [we already saw one family of examples]
Mathematically this equivalence $\Longleftrightarrow$ Existence of projectively flat unitary connection on the quantum bundle with parallel transport establishing the equivalence of quantizations

* Q2// What if we restrict $\mathcal{J}$ to natural smaller families?

A2// Then the answer becomes yes in some cases

Example 1 [Axelrod - Della Pietra - Witten, Geometric quantization of CS gauge theory, J Diff Geom 33 (1991)]

$$z = q + \tau p \qquad \mathbb{R}^{2n} \qquad (3.6)$$

$$\tau \in \check{J}_{TR} \simeq \mathbb{H}_n = \{\tau \in Mat_n(\mathbb{C}), \tau^T = \tau, \ \mathrm{Im}\,\tau > 0\} \cong Sp(2n,\mathbb{R})/U(n)$$

$\mathcal{H}^Q_{TR}$

$\downarrow$

$\check{J}_{TR}$

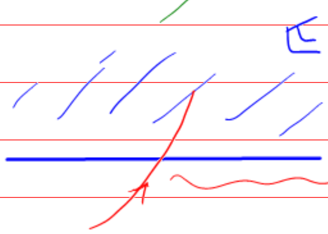
These are translation invariant Kähler polarizations (quantizations) and for them there is a natural unitary projectively flat connection on the restricted quantum bundle establishing the equivalence of these quantizations.

Its (projectively) unitary parallel transport is given by the lifting of

$Sp(2n,\mathbb{R})$ ($Mp(2n,\mathbb{R})$ if we take into account the half form correction)

from the space of translation-invariant Kähler polarizations to the quantum bundle.

$\mathbb{H}_n = Sp(2n, \mathbb{R}) / U(n)$



To cross from pseudo-Kähler polarizations to Kähler need complex symplectomorphisms

Example 2

T^*K [Hall JFA 1994, Thiemann CQG 1996, Ashtekar-Lewandowski-Marolf-M-Thiemann JFA 1996, Hall CMP 2002, Florentino-Matias-M-Nunes JFA 2005, 2006, ...]

K - compact Lie group (or of compact type to include $\mathbb{R}^n$)

$$T^*K \cong K \times \mathfrak{k}^* \cong K \times \mathfrak{k} \cong K^{\mathbb{C}}$$

$$(x, Y^*) \mapsto (x, \tau Y) \mapsto x e^{\tau Y} \qquad (3.7)$$

$$\psi_\tau : T^*K \xrightarrow{\sim} K^{\mathbb{C}}$$

Theorem [Lempert-Szöke 1991, Guillemin-Stenzel 1992, Thiemann 1996, Hall-Kirwin 2011]

(3.8)

For any $\tau \in \mathbb{H}_1$ the observables

$$\{\pi_{ij}(x e^{\tau Y})\} \sim \mathcal{P}_\tau \qquad \pi \in \hat{K} = \text{irrep}\, K/_\sim$$

define a Kähler structure on T^*K.
This family of Kähler polarizations/quantizations can be obtained by applying the geodesic flow to the Schrödinger polarization/quantization defined by the observables AND continuing to time $\tau \in \mathbb{H}_1$

$$H = \frac{1}{2}\|y\|^2 = \frac{1}{2}\sum_{jk} g^{jk}(x) P_j P_k \tag{3.9}$$

$$H \rightsquigarrow X_H \rightsquigarrow \varphi_t^{X_H} = e^{tX_H}$$

$$\rightsquigarrow e^{\tau X_H}$$

Theorem [Hall 2002, FMMN 2005, 2006, Kirwin-M-Nunes, 2013]
The lifting of the above $(\mathbb{H}_1, +)$ action to the quantum bundle is unitary, is equivalent to the Hall Coherent State Transform (CST) and to the BKS pairing map from the

Schrödinger quantization to the Kähler quantization.

* These two examples "justify" why we don't hear about the infinite dimensional space of quantizations of ANY system: The most common systems have preferred polarizations/quantizations (even families of unitarily equivalent ones as in the examples) so that one is usually satisfied with that.

* BUT in nontrivial systems as is the case of Yang-Mills theory General Relativity or with finite dimensional physical phase-space, Chern-Simons theory one may be forced to consider observables which have a complicated relation with canonical pairs of observables.

* For example in T^*K instead of the family of polarizations considered above we consider the following infinite dimensional family of Kähler Polarizations,

$$\pi_{jk}\left(x\, e^{\tau \frac{\partial H}{\partial y}}\right) \qquad \tau \in \mathbb{H}, \tag{3.10}$$

$H(Y) \rightsquigarrow$ Ad_K - invariant convex function on $\mathfrak{h}$

Poisson brackets: $\left(H_0(Y) = \frac{1}{2}\|Y\|^2 \text{ in the usual } cST \text{ case}\right)$

$\{y_a, f(x)\} = (X_a f)(x)$ X_a - left invariant vector field

Thm The natural extension of Hall CST is not unitary if H is not quadratic. [Kirwin - M - Nunes, 2013, 2014]

Note: We will return to this example in example 4 and in Lecture 4.

3.2 Geometry on the space of quantizations

Fortunately to our rescue there is an ∞-dimensional "group" which "acts" on the space of quantizations of (M,ω), $\mathcal{J}(M,\omega)$, with essentially finite codimension orbits and mixing polarizations of DIFFERENT TYPES

Non existing complexi-fication ... see Rmk below

$$G_{\mathbb{C}} = \text{"}Ham_{\mathbb{C}}(M,\omega)\text{"} \qquad (3.11)$$

where $G = Ham(M,\omega)$, with $Lie(G) = (C^\infty(M), \{,\})/\mathbb{R}$ is an infinite dimensional analogue of a compact Lie group with Ad invariant positive definite inner product on its Lie algebra given by

$$\langle f, g \rangle = \int_M f g \frac{\omega^n}{(2\pi h)^n} \qquad (3.12)$$

The orbits of G on the space of quantizations $\mathcal{J}$ are expected to lead to equivalent quantizations so that the challenge is to study the action of imaginary time 1-parameter subgroups

$$e^{itX_f} \circlearrowright \mathcal{J} \qquad (3.13)$$

and the quotient

$$G_{\mathbb{C}}/G \qquad (3.14)$$

in general

RMK Even though the group $G = \mathrm{Ham}(M,\omega)$ does not have a complexification we can, following Donaldson, study spaces which play the Role of its orbits.

Let (ω, J_0) be a Kähler pair and consider then

$$\underbrace{\mathcal{H}(\omega, J_0)}_{\boxed{\text{very important also in Kähler geometry}}} = \left\{ \begin{array}{l} \text{Kähler metrics on } (M, J_0) \text{ in} \\ \text{the same cohomology class} \\ \text{of } \omega \end{array} \right\}$$

if M is compact

$$= \{\omega_f = \omega + i\partial_0\bar{\partial}_0 f,\ f \in C^\infty(M) : \omega_f > 0\} \qquad (3.15)$$

$$\underset{\text{Moser Thm}}{\cong} \{\varphi^* \omega,\ \underbrace{\varphi \in \mathrm{Diff}(M) : (\varphi^*\omega, J_0) \text{ is Kähler}}_{= G_{\mathbb{C}}}\} = G_{\mathbb{C}}^* \omega \cong G_{\mathbb{C}}/G$$

It is natural to identify this subset of $\mathrm{Diff}(M)$ as $G_{\mathbb{C}}$ because

$$L_{J_0 X_f^\omega}\, \omega = \frac{i}{2}\partial_0\bar{\partial}_0 f \qquad (3.16)$$

Furthermore

$$\mathcal{H}(\omega, J_0) = G_{\mathbb{C}}^* \omega \cong G_{\mathbb{C}}/G \qquad (3.17)$$

has the structure of a symmetric space with negative curvature for the Mabuchi metric

$$\langle f_1, f_2 \rangle_{\omega_f} = \int_M f_1 f_2\, \omega_f^n \qquad (3.18)$$

with geodesics given by the analogue of imaginary time one parametre subgroups of $G_{\mathbb{C}}$.

In geometric quantization rather than fixing the complex structure and acting on the symplectic form

$$(G_{\mathbb{C}}^{*}\omega, J_0) \cong \mathcal{H}(\omega, J_0) \quad (3.19)$$

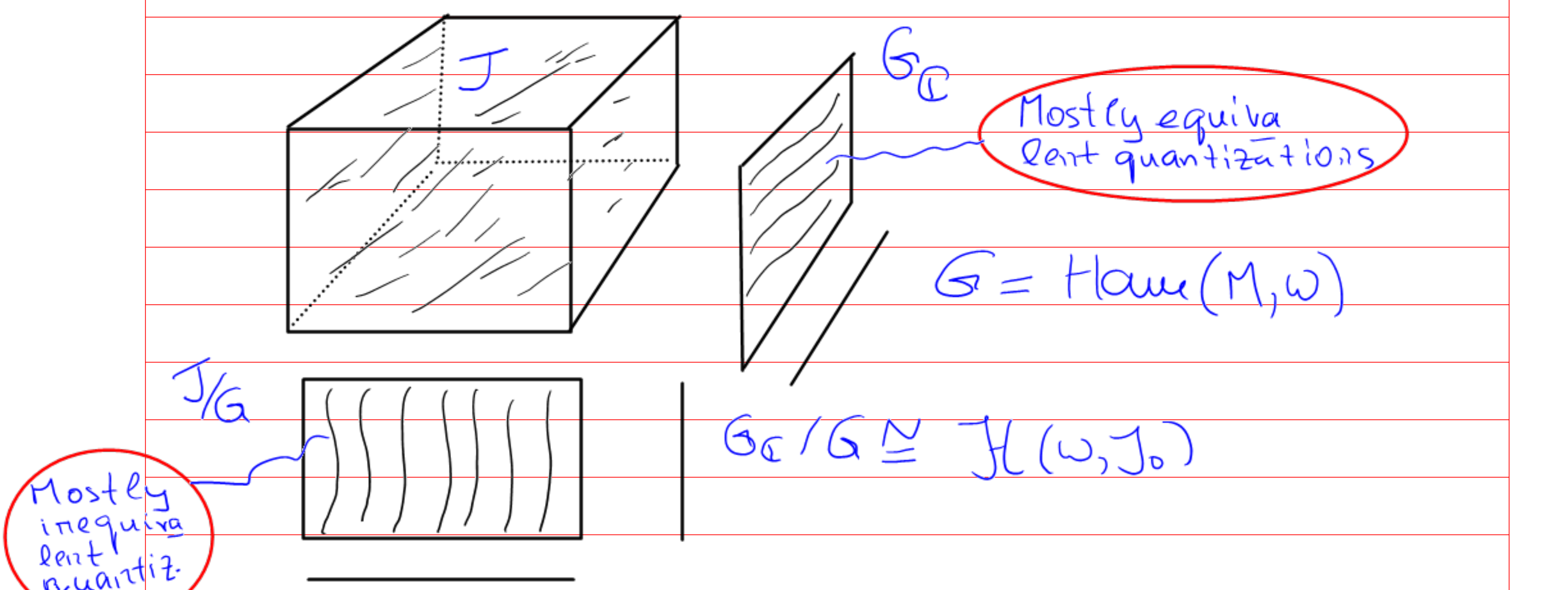
is more interesting to fix the symplectic form and act on the complex structure. Assuming, for simplicity, that J_0 has no symmetries we get

$$G_{\mathbb{C}} \cdot J_0 \cong G_{\mathbb{C}}$$

$$(\omega, G_{\mathbb{C}} J_0) \longrightarrow (G_{\mathbb{C}}\,\omega, J_0)$$

$$(\omega, \varphi_* J_0) \longmapsto (\varphi^{*}\omega, J_0) \quad (3.20)$$

$$\mathcal{J} \supset G_{\mathbb{C}} \longrightarrow G_{\mathbb{C}}/G \cong \mathcal{H}(\omega, J_0)$$



$\mathcal{J}/G_{\mathbb{C}} \cong \mathcal{M}$ = Moduli space of cx structures

3.3 On the explicit construction of geodesics in the space of Kähler metrics / Kähler quantizations [M-Nunes, 2013]

So we will be studying equivalence of quantizations $\mathcal{J}$ along special one parameter families corresponding to geodesics in $\mathcal{J}$

$$\mathcal{P}^{h}_{it} = e^{itL_{X_h}}\mathcal{P}_0 \qquad (3.21)$$

From (3.15) we see that the challenge we faced to get explicit expressions for the geodesics was, after defining (3.21) to map back the flow to Diff(M) [Burns-Lupercio-Uribe, 2013, developed before [MN] an alternative approach]

* The algorithm is simple:

(see next section for complex h)

Let $h \in C^\infty(M)$. and (ω, J_0) be Kähler

(i) Find the flow of X_h, $\varphi_t^{X_h}$ and extend analytically to complex time

flow

$$\varphi_\tau^{X_h} \qquad (3.22)$$

(ii) act on (local) J_0-holomorphic functions

Restrict

$$z_k^{(\tau)} := \left(\varphi_\tau^{X_h}\right)^*(z_k) \qquad (3.23)$$

(iii) For sufficiently small $|\tau|$ $\exists!$ a diffeomorphism

extend

$\tilde{\varphi}_\tau^{X_h, J_0}$: $\quad J_\tau = \left(\tilde{\varphi}_\tau^{X_h, J_0}\right)^* J_0 \quad (3.24)$

is the complex structure with J_τ-holomorphic coordinates $z_k^{(\tau)}$

* The explicit form for the Kähler metric / form is then [MN 2013]

$$\gamma_\tau(X, Y) = \omega(J_\tau X, Y) \qquad (3.25)$$

* The corresponding Kähler potential solving the geodesic eqs. for the Mabuchi metric on $\mathcal{H}(\omega, J_0)$

Homogeneous Monge-Ampère equation

$$\ddot{K}_t = \frac{1}{2} \| \nabla \dot{K}_t \|^2_{\gamma_{it}} \qquad (3.26)$$

is given by the following expression

$$K_t = \operatorname{Re} \Psi_{it} \qquad (3.27a)$$

where

$$\Psi_{it} = \left(\varphi_{it}^{X_h}\right)^* K_0 + 2th - 2i\,\alpha_{it} \qquad (3.27b)$$

and α_{it} is the analytic continuation to the imaginary axis of

$$\alpha_t = \int_0^t e^{t' X_h} \theta(X_h)\, dt' \qquad (3.27c)$$

The map above

$$\{\varphi_\tau^{X_f},\ |\tau|<T\} \to \widetilde{\mathrm{Diff}}(M)$$

$$\varphi_\tau^{X_f} \longmapsto \tilde{\varphi}_\tau^{X_f,\mathcal{P}_0} \qquad (3.28)$$

is not a group homomorphism but (a (local) action groupoid of the one parameter group $\{\varphi_\tau^X, \tau\in\mathbb{C}\}$ on the space of complex structures (?) NO (!) on the space of polarizations

$$\tilde{\varphi}_{\tilde{\tau}}^{X_f,\mathcal{P}_\tau} \tilde{\varphi}_\tau^{X_f,\mathcal{P}_0} = \tilde{\varphi}_{\tau+\tilde{\tau}}^{X_f,\mathcal{P}_0} \qquad (3.29)$$

complex time

complex Hamilt.

By considering $f = f_1 + i f_2$ we can repeat the steps (i), (ii), (iii) above to get (see section 3.4 and (3.31)-(3.34) below)

$$\varphi_t^{X_f} \rightsquigarrow \tilde{\varphi}_t^{X_f,\mathcal{P}_0}$$

Then

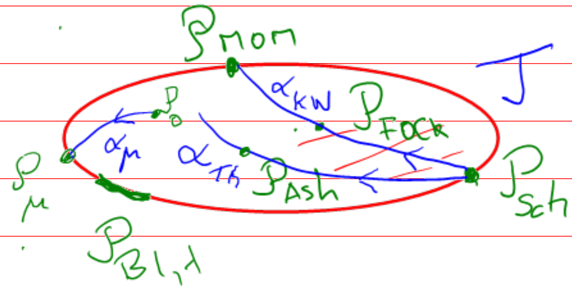
Theorem [Burns-Lupercio-Uribe]

$\tilde{\varphi}_t^{X_f,\mathcal{P}_0}$ is the solution of the following problem

$\mathcal{P}_0 = \mathcal{P}_{J_0}$

$$\underbrace{\begin{cases} \dot{y}_t = X_{f_1} + \nabla^{\gamma_t} f_2 \\ \gamma_t(X_1, X_2) = \omega\left(\left(\tilde{\varphi}_t^{X_f,\mathcal{P}_0}\right)^* (J_0)(X_1), X_2\right) \end{cases}}_{\text{Dynamics generated by one complex Hamiltonian in the original phase space}} \qquad (3.30)$$

3.4 Examples of Geodesics in the space of quantizations



In the previous section we described a simple algorithm to reduce the equation for geodesics in the space of quantizations (the Cauchy problem) = non linear PDE for the Kähler potential (3.26) to the classical Hamilton eqs continued to imaginary time (see also (3.27))

* Since our goal in section 4 will consist in lifting these geodesics to the quantum bundle in order to study the equivalence of quantizations along them we will now describe two illustrative examples of this method.

* RMK This method is usefull also in Kähler geometry as a method of reducing the non linear Monge Ampère equation (3.26) to the classical Hamiltonian eqs.

* Let us adapt the algorithm to the more general situation of a complex valued Hamiltonian as the results are of interest also in the context of PT symmetric quantum mechanics

* Let $h \in C^\infty(M,\mathbb{C})$ and (ω, J_0) be Kähler
$h = h_1 + i h_2$

(i) Find the action of $\varphi_t^{X_h} = e^{tX_h}$ on real analytic complex valued functions on M. (3.31)

flow

(ii) Restrict the action of $\varphi_t^{X_h}$ to local J_0-holomorphic functions

Restrict

$$z_k^{(t)} := e^{tX_h}(z_k) \tag{3.32}$$

(iii) For sufficiently small $|t|$ $\exists!$ a diffeomorphism

Extend

$$\tilde{\varphi}_t^{X_h, P_0} : \quad J_t = \left(\tilde{\varphi}_t^{X_h, P_0}\right)^* J_0 \tag{3.33}$$

J_t is the complex structure with J_t-holomorphic coordinates $z_k^{(t)}$
In local coordinates this diffeomorphism is given by

$$\tilde{\varphi}_t^{X_h, P_0} : (z_k^{(t)}, \bar{z}_k^{(t)}) = \left(e^{tX_h} z_k, e^{tX_{\bar{h}}} \bar{z}_k\right) \tag{3.34}$$

Example 3 $\mathbb{R}^2$, $h = iP^2/2$

Let $(M,\omega) = (\mathbb{R}^2, dq \wedge dp)$, $h = \frac{iP^2}{2}$ so that we are trying to describe free motion but in imaginary time. Let us start at $\mathcal{P} = \mathcal{P}_{Fock} = \langle X_z \rangle$, $z = q + ip$
As we saw in (3.30) this corresponds to solving the following system

$$X_h = ip\frac{\partial}{\partial q}$$

$$\tilde{\varphi}_t^{X_{h_1},p_0} : \begin{cases} \begin{pmatrix} \dot q \\ \dot p \end{pmatrix} = X_{h_1} + \nabla^{\gamma_t} h_2 = \nabla^{\gamma_t} \frac{p^2}{2} \\ \gamma_t(X,Y) = \omega\left(\left(\left(\tilde{\varphi}_t^{X_{h_1},p_0}\right)^*(J_0)\right)(X), Y\right) \end{cases} \tag{3.35}$$

If the metric was not changing imaginary time free motion would just be a dilation of momenta

$$\begin{cases} \nabla^{\gamma_0} h_2 = p\frac{\partial}{\partial p} \\ \begin{pmatrix} \dot q \\ \dot p \end{pmatrix} = \begin{pmatrix} 0 \\ p \end{pmatrix} \qquad (q^{(t)}, p^{(t)}) = (q, e^t p) \end{cases} \tag{3.36}$$

In our case we will see that

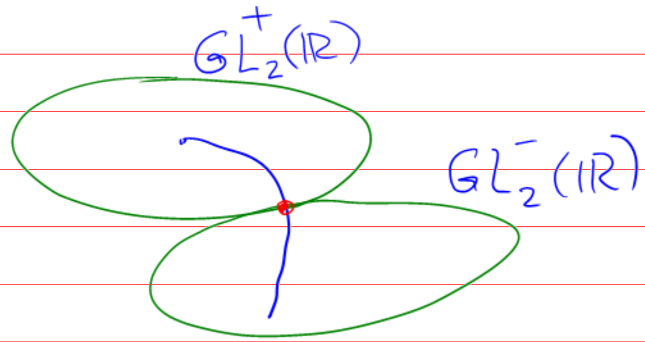
$$\left\{ e^{itp\frac{\partial}{\partial q}}, t \in \mathbb{R} \right\} \longrightarrow \left\{ \tilde{\varphi}_t^{X_{h_1},p_0}, t \in \mathbb{R} \right\} \subset \widetilde{GL_2}(\mathbb{R}) \tag{3.37}$$

will take us in a journey from $GL_2^+(\mathbb{R})$ to $GL_2^-(\mathbb{R})$ through a strange boundary point

$$\tilde{\varphi}_{-1}^{X_{h_1},p_0} : \mathbb{R}^2 \to \mathbb{R}^2 \tag{3.38}$$

$$\tilde{\varphi}_{-1}^{X_{h_1},p_0}(q,p) = (q,0)$$

$LHam_{\mathbb{C}}(\mathbb{R}^2,\omega) \longrightarrow$



(3.39)

Indeed, applying the algorithm (3.31)-(3.34) we get

(3.40)

(i) $e^{tX_h}(q,p) = e^{itp\frac{\partial}{\partial q}}(q,p) = (q+itp, p)$

(ii) Restricting (i) to J_0-holomorphic functions

(3.41)

$$e^{tX_h} z = e^{itp\frac{\partial}{\partial q}}(q+ip) =$$

$$= q+itp+ip = q+i(t+1)p = z^{(t)}$$

(iii) We see that the unique diffeo mapping (for small t)

$$z \longmapsto z^{(t)} = \left(\tilde{\varphi}_t^{X_h, \mathcal{P}_0}\right)^*(z)$$

(3.42)

is

$$\tilde{\varphi}_t^{X_h,\mathcal{P}_0}(q,p) = (q, (t+1)p)$$

We see that $\tilde{\varphi}_t^{X_h,\mathcal{P}_0}$ is indeed a rescaling of p which is not a one parameter subgroup of $GL_2(\mathbb{R})$ and behaves as predicted in (3.38). Let us show that it satisfies the system (3.35). Let us first find the metric. It is easier to find first ω_t (which is just the pull back) in the $\mathcal{P}_t$ coordinates $z^{(t)}, \bar{z}^{(t)}$ as the metric has the same coefficient function(s) in the corresponding frames

$$z^{(t)} = q + i(t+1)p$$

(3.43)

Ex
$$\Rightarrow \quad \omega = -\frac{i}{2}\frac{dz^{(t)} \wedge d\bar{z}^{(t)}}{1+t} \Rightarrow$$

Ex
$$\Rightarrow \quad \gamma_t = \frac{1}{1+t}dq^2 + (1+t)dp^2$$

This is the solution of the second

equation in (3.35). For the first one we have

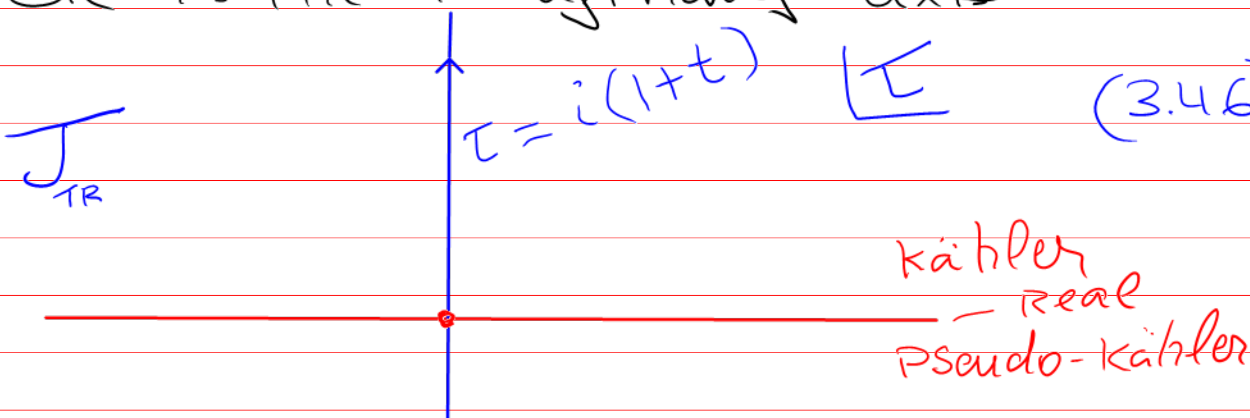
$$\nabla^{\gamma_t} \frac{p^2}{2} = \frac{1}{1+t}\, p \frac{\partial}{\partial p} \qquad (3.44)$$

and therefore

$$\begin{cases} \dot{q} = 0 \\ \dot{p} = \dfrac{1}{1+t}\, p \end{cases} \qquad (3.45)$$

$$\Rightarrow \quad (q^{(t)}, p^{(t)}) = (q, (1+t)p)$$

In the space of translation-invariant polarizations/quantizations our geodesic is the imaginary axis



(3.46)

$$\mathcal{P}_\tau = \langle X_{z^\tau} \rangle$$

$$z^\tau = q + \tau p$$

$$\mathcal{P}_{\tau=0} = \mathcal{P}_{t=-1} = \mathcal{P}_{Sch}$$

Example 4 T^*K, $h = iH(y)$, Ad_K-invariant and convex

Let ω be the canonical symplectic form and $\mathcal{J}_0$ be the kähler polarization for which the holomorphic functions are (as in (3.10))

$$\pi_{jk}\left(x e^{i \frac{\partial H}{\partial y}}\right) \tag{3.47}$$

We have

$$\text{i)}\quad X_h = i \sum_j X_j \frac{\partial H}{\partial y_j}$$

and

$$\text{ii)}\quad e^{tX_h}\, \pi_j\left(x e^{i\frac{\partial H}{\partial y}}\right) = \pi_j\left(x e^{i(1+t)\frac{\partial H}{\partial y}}\right) \tag{3.48}$$

So that, similarly to example 1,

$$\text{iii)}\quad \tilde{\varphi}_t^{X_h, \mathcal{P}_0}(x, y) = (x, y_t) \tag{3.49}$$

where

$$y_t = L^{-1}\left((1+t) L(y)\right)$$

and $L(y) = \dfrac{\partial H}{\partial y}$.

From the expression (2.48) for the $\mathcal{J}_t$-holomorphic functions we see that the situation is very similar to the flat case. In particular $\mathcal{P}_{-1} = \mathcal{P}_{Sch}$ and

$$\tilde{\varphi}_{-1}^{X_h, \mathcal{J}_0} : T^*K \longrightarrow T^*K$$
$$(x, y) \longmapsto (x, 0)$$

RMK So $\tilde{\varphi}_{-1}^{X_h,\mathcal{P}_0}$ collapses T^*K to K but being a grupoid morphism should be invertible...

So what is $(\tilde{\varphi}_{-1}^{X_h,\mathcal{P}_0})^{-1}$??

we should look at $\tilde{\varphi}^{X_h,\mathcal{P}_0}$ as acting primarily on polarizations and see a posteriori what this corresponds to in terms of the manifold via the polarized functions. So we saw that

$\boxed{f: X(f)=0 \ \forall X \in \Gamma(\mathcal{P})}$

$$\tilde{\varphi}_{-1}^{X_h,\mathcal{P}_0} : \mathcal{P}_{sch} \longrightarrow \mathcal{P}_0$$

or, in terms of polarized functions,

$$\left(\tilde{\varphi}_{-1}^{X_h,\mathcal{P}_0}\right)^*$$ pull backs $\mathcal{P}_0$ holomorphic functions on $T^*K \cong K_{\mathbb{C}}$ to (the totally real submanifold) K.

Its inverse then should be

$$\left(\tilde{\varphi}_{-1}^{X_h,\mathcal{P}_0}\right)^{-1} = \tilde{\varphi}_{1}^{X_h,\mathcal{P}_{sch}} : \mathcal{P}_0 \longrightarrow \mathcal{P}_{sch}$$

or in terms of polarized functions analitically continue to $K_{\mathbb{C}}$ real analytic functions on K which is precisely what e^{tX_h} does for all positive t including $t=1$ as was the goal of complexifiers introduced by Thiemann in 1996.

$$e^{-i\sum_j \partial_j H(Y) X_j} \quad \boxed{\begin{matrix}\text{collapse}\\ T^*K \to K\end{matrix}}^{-1} = \boxed{\begin{matrix}\text{Analytic continuation}\\ K \hookrightarrow T^*K\end{matrix}} \quad e^{+i\sum_j \partial_j H(Y) X_j}$$