Analysis of complex singularities in high-Reynolds-number Navier-Stokes solutions

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joint work with

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Introduction

The interaction of an incompressible fluid at high Reynolds number with a rigid boundary is one of the main interest phenomena in the classical fluid dynamics theory.

It is in fact known as such interaction is one of the possible mechanisms of the transition from laminar to turbulent regimes.

- Introduction to Boundary Layer Theory
  - Prandtl equation for the “impulsively started” disk.
  - Singularity formation.
- Comparison between Prandtl and Navier-Stokes solutions
  - Large-scale and small-scale phenomena.
- Complex Singularity Tracking
  - Padè approximants and Pólya method.
- Work in progress

Navier-Stokes Equations

$$\begin{aligned}\frac{\partial \mathbf{u}^{NS}}{\partial t} + \mathbf{u}^{NS}\cdot\nabla\mathbf{u}^{NS} + \nabla p &= \frac{1}{Re}\Delta\mathbf{u}^{NS}\\ \nabla\cdot\mathbf{u}^{NS} &= 0,\\ \mathbf{u}^{NS}(\mathbf{x},t=0) &= \mathbf{u}_0.\end{aligned}$$

Euler Equations

$$\begin{aligned}\frac{\partial \mathbf{u}^{E}}{\partial t} + \mathbf{u}^{E}\cdot\nabla\mathbf{u}^{E} + \nabla p &= 0\\ \nabla\cdot\mathbf{u}^{E} &= 0,\\ \mathbf{u}^{E}(\mathbf{x},t=0) &= \mathbf{u}_0.\end{aligned}$$

Without boundaries,

$$\left\|\mathbf{u}^{NS} - \mathbf{u}^{E}\right\| \to 0 \quad \text{for} \quad Re \to +\infty,$$

- Swann *Trans AMS* 1971 in $\mathbb{R}^3$,
- Constantin & Wu *Nonlinearity* 1995 in $\mathbb{R}^2$ for initial data of "vortex patch" type.

Navier-Stokes equations

$$\begin{aligned}
\frac{\partial \mathbf{u}^{NS}}{\partial t} + \mathbf{u}^{NS} \cdot \nabla \mathbf{u}^{NS} + \nabla p &= \frac{1}{Re} \Delta \mathbf{u}^{NS} \\
\nabla \cdot \mathbf{u}^{NS} &= 0, \\
\mathbf{u}^{NS}|_{\partial\Omega} &= 0, \\
\mathbf{u}^{NS}(\mathbf{x}, t = 0) &= \mathbf{u}_0.
\end{aligned}$$

Euler Equations

$$\begin{aligned}
\frac{\partial \mathbf{u}^{E}}{\partial t} + \mathbf{u}^{E} \cdot \nabla \mathbf{u}^{E} + \nabla p &= 0 \\
\nabla \cdot \mathbf{u}^{E} &= 0, \\
\nu \cdot \mathbf{u}^{E}|_{\partial\Omega} &= 0, \\
\mathbf{u}^{E}(\mathbf{x}, t = 0) &= \mathbf{u}_0.
\end{aligned}$$

- The different number of BC generates a Boundary Layer which expands to the internal flow, due to non-linearity.

To study the Boundary Layer flux, Prandtl (1904) introduced the following scaling, valid near the boundary:

$$Y = y\sqrt{Re} \qquad \text{with} \qquad \frac{\partial u}{\partial Y} = O(1).$$

This implies that:

$$\frac{\partial u}{\partial x} = O(1), \qquad \frac{1}{Re}\frac{\partial^2 u}{\partial y^2} = \frac{1}{Re}\frac{\partial^2 u}{\partial Y^2} = O(1) = u\frac{\partial u}{\partial x}. \tag{1}$$

Prandtl Equations

$$\begin{aligned}
\partial_t u^P + u^P \partial_x u^P + v^P \partial_Y u^P + \partial_x p &= \partial_{YY} u^P \\
\partial_Y p &= 0 \\
\partial_x u^P + \partial_Y v^P &= 0 \\
u^P(x, Y = 0) = v^P(x, Y = 0) &= 0 \\
u^P(x, Y \to \infty) \longrightarrow & \ u^E(x, y = 0) \\
u^P(x, y, t = 0) &= u_{in} \ .
\end{aligned}$$

The procedure is the following: first solve Euler equations, to obtain the boundary data $\mathbf{u}^E(y = 0)$ and then solve Prandtl equation.

$$\text{Conjecture :} \quad \mathbf{u}^{NS} = m\left(y\sqrt{Re}\right)\mathbf{u}^E + \left(1 - m\left(y\sqrt{Re}\right)\right)\mathbf{u}^P + O\left(Re^{-\frac{1}{2}}\right).$$

Kato (1984)

The following are equivalent:

- $$\lim_{Re\to+\infty}\int_0^T \frac{1}{Re}\int_{\{d(\mathbf{x},\partial\Omega)<\frac{c}{Re}\}} \left|\nabla \mathbf{u}^{NS}\right|^2 dxdt = 0.$$
- $$\left\|\mathbf{u}^{NS}-\mathbf{u}^{E}\right\| \to 0 \quad \text{per} \quad Re\to+\infty \quad \text{uniform} \quad \text{in} \quad t\in[0,T]\,.$$

If there is no energy anomalous dissipation at the boundary, in a layer of amplitude $1/Re$, and then in the entire domain, then the zero viscosity limit presents no turbulent behaviour.

If you want to solve the problem of the zero viscosity limit of the NS equations must be checked or improve (or disprove) Prandtl equations.

Well posedness:

- Oleinik 1967 ($\partial_Y \mathbf{u}_{in} > 0$).
- Xin and Zhang '03 (favourable pressure gradient).
- Sammartino and Caflisch 1998 (analytic initial data).
- Lombardo, Cannone and Sammartino '03, '13 (analytic initial data in the streamwise direction).

Singularities for Prandtl equation:

- Van Dommelen and Shen '80:

show numerically, with Lagrangian methods, that Prandtl solutions developed shock type singularities at a finite time

These results are confirmed in Cowley and Van Dommelen '90, o Hong e Hunter '04.

- E and Engquist '97:

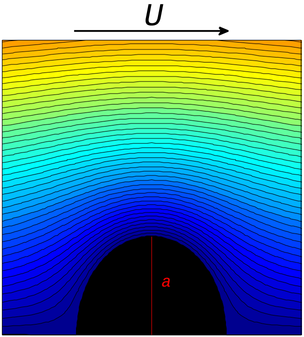
prove analytically, that for suitable initial data, different from VDS data, that Prandtl solutions developed a shock type singularity.

- Gargano, Sammartino and S. Physica D '09:

using the complex singularity tracking method, characterize the Prandtl singularity and they give the first numerical evidence that Prandtl equations are ill-posed in H^1.

VDS datum

Consider the Prandtl and Navier-Stokes equations in the case of a disk in a uniform flux, impulsively started. The physical domain is $[\theta, r]$=$[0, \pi]\times$[a,∞], where a is the radius of the disk.



The inviscid irrotational solution of the Eulero equations, given in terms of the streamfunction ($\psi_x = -v, \psi_y = u$), is the following:

$$\psi(\theta, r) = U(r - \frac{a^2}{r}) \sin(\theta),$$

where U is the streamwise component of the velocity at infinity.

VDS datum: Prandtl equation

In the case of the impulsively started disk, the equation of Prandtl are the following:

VDS Datum

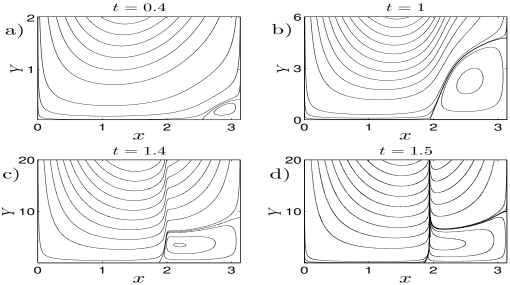
$$\begin{aligned} \partial_t u + u\partial_x u + v\partial_Y u - U\partial_x U &= \partial_{YY} u \qquad [0,\pi]\times[0,\infty] \\ \partial_x u + \partial_Y v &= 0 \\ u(x,0,t) = v(x,0,t) &= 0 \\ u(x,\infty,0) &= U \\ u(x,y,0) &= U \\ U = 2\sin(x) & \end{aligned}$$

The solution develops a **singularity** in a finite time

(Van Dommelen & Shen *J.Comp.Phys.* 80').

To solve numerically the equation of Prandtl we used a fully-spectral Fourier-Chebyshev (Pseudo–Spectral τ–method) numerical method and a RK2CN for the advance in time.

VDS datum: Prandtl equation



The formation of VDS singularity is due to the phenomenon of recirculation:

- Formation of a back-flow, and of a "stagnation point", due to a adverse pressure gradient at time $t \approx 0.4$;
- At $t \approx 1$ two "counter-rotating" vortex are visible;
- The two vortex grow forming a "kink" at $t \approx 1.35$ which evolves in a "sharp spike" at $t \approx 1.5$;
- The vorticity in the Boundary Layer is expelled to the external flow, and the normal component of the velocity becomes infinite (in the BL scale) in the streamwise location of the singularity: separation phenomenon.

VDS datum: NS equations

The Navier-Stokes equations in the vorticity-streamfunction form:

Vorticity-Streamfunction system

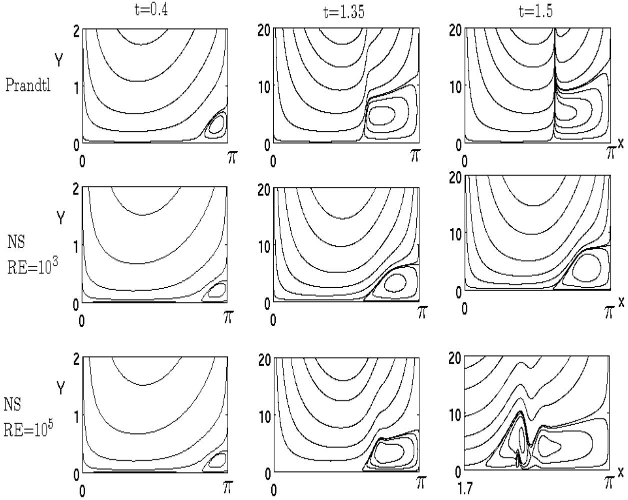
$$
\begin{aligned}
\frac{\partial \omega}{\partial t} + \frac{u}{r}\frac{\partial \omega}{\partial \theta} + v\frac{\partial \omega}{\partial r} &= \frac{1}{Re}\Delta_{\theta,r}\omega, \qquad [0,\pi]\times[1,\infty] \\
\Delta_{\theta,r}\psi &= -\omega, \\
u = \frac{\partial \psi}{\partial r}, v &= -\frac{1}{r}\frac{\partial \psi}{\partial \theta}, \\
\omega(\theta, r, t=0) &= 0, \\
\omega(\theta, r\to\infty, t) &\to 0, \\
u(\theta, r=1, t) &= v(\theta, r=1, t) = 0.
\end{aligned}
$$

where $\Delta_{\theta,r}$ is the Laplacian in cylinder coordinate, and the *Reynolds* number is

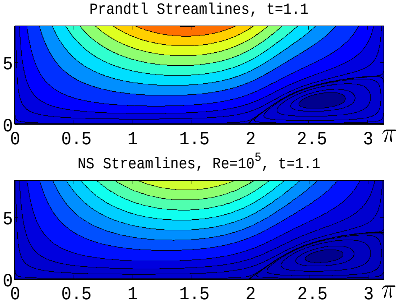
defined as $Re = \frac{aU}{\nu}$.

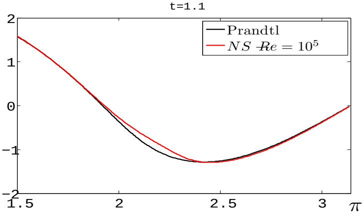
To solve numerically NS equations, for different Re (from $Re = 10^3$ at $Re = 10^5$), we used a Fourier-Chebyshev fully-spectral Galerkin-Collocation numerical method with an AD-BDI2 (with the influence matrix method to compute the BC).

The streamline for Prandtl and NS at different Re.



Large-Scale interaction



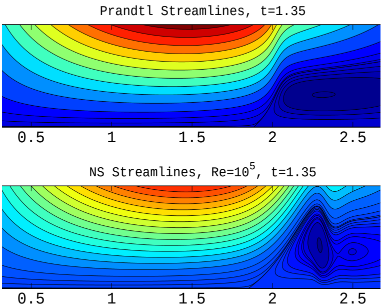


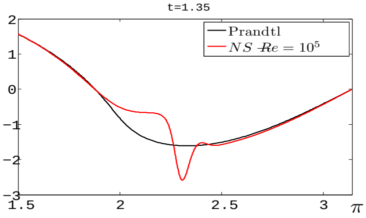
"Wall shear stresses" (the vorticity at the boundary):

- $\tau_w^P = \partial_Y u^P_{|Y=0}$ (Prandtl)
- $\tau_w^{NS} = Re^{-1/2} \partial_y u^{NS}_{|y=0}$ (Navier-Stokes)

- The LS interaction appears for all the Re:
  The evolution of the flow is similar to that predicted by the BL equation, it is visible a single region of recirculation. However, one can notice some initial quantitative differences between Prandtl and NS solutions.

Small-Scale interaction





"Wall shear stresses" (the vorticity at the boundary):

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- $\tau_w^{NS} = Re^{-1/2} \partial_y u^{NS}_{|y=0}$ (Navier-Stokes)

For "moderate-high" Re ($Re \geq O(10^4)$) the LS interaction evolves to a small-scale interaction, characterize:

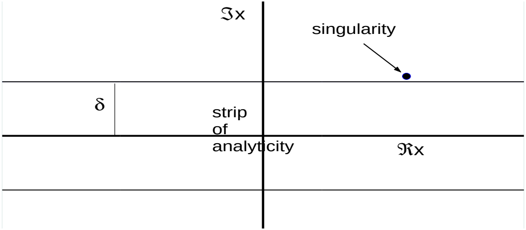
- Splitting of the recirculation region.
- Formation of high gradients in the angular direction.

Complex Singularity Tracking

Let $u(Z) = (Z - Z_*)^\alpha = \sum_{k=-\infty}^{\infty} \hat{u}_k e^{ikZ}$, be an analytic function with a singularity of algebraic type α in $Z_* = x^* + i\delta$, then the asymptotic behaviour of its Fourier coefficients is given by the following (Laplace formula):

$$\hat{u}_k \sim C|k|^{-(1+\alpha)} \exp(-\delta k) \exp(ix^* k).$$

The exponential decay rate of the spectrum δ gives the width of the analyticity strip.



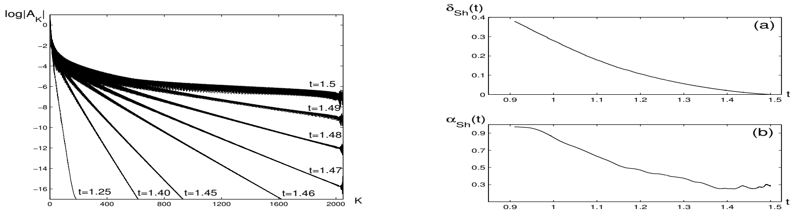
If the complex singularity reaches the real axis, the singularity is shown in the "real world" as a blow up (of the solution or of its derivatives).
The singularity time t_s is the time when $\delta(t_s) = 0$. x^* and α give, respectively, the real location and the algebraic character of the singularity.
(Sulem, Sulem & Frisch'83, Frish et al., Caflisch, Cowley, Pugh, Shelley, Tanveer)

Complex Singularity Tracking

Given a function $u(z,w) = \sum_{h,k} a_{hk} e^{ihz} e^{ikw}$, one defines the "shell–summed Fourier amplitude" (Frisch et al. '05) as

$$A_K \equiv \sum_{K \le |(h,k)| < K+1} a_{hk}.$$

Using the asymptotic Laplace formula for A_K, one can determines the distance δ of the complex singularity and its algebraic type α.



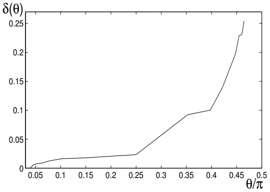
Application to Prandtl equation in Della Rocca, Lombardo, Sammartino, S. J. App. Num. Math. '05, Gargano, Sammartino, S. Physica D '09.

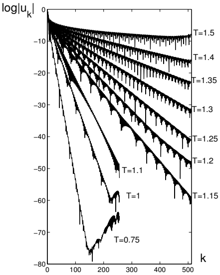
Complex Singularity Tracking

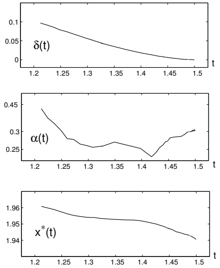
Another possible method (Poincaré 1899, Tsikh 1993), consists to evaluate the asymptotic Laplace formula to each direction of the bi-dimensional spectrum,

$$(h, k) = K(\cos(\theta), \sin(\theta)).$$

The distance δ is the minimum between all directions.



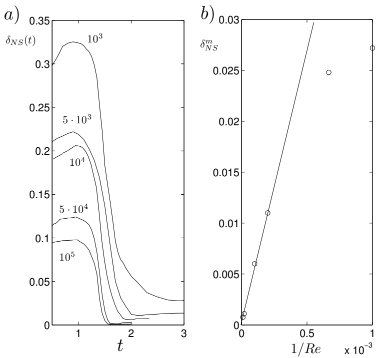


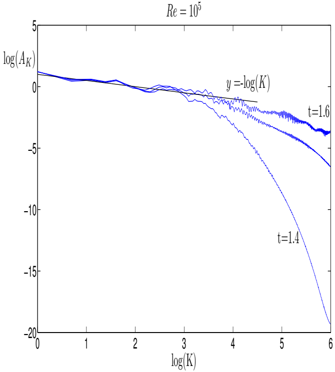


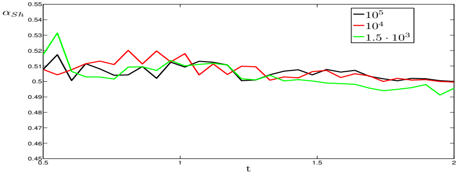
Application to Prandtl equation Della Rocca, Lombardo, Sammartino, S.J. App. Num. Math. '05, Gargano, Sammartino, S.Physica D '09

Complex Singularity Tracking

Application to NS Gargano, Sammartino, S., Cassel J. Fluid Mech. '14







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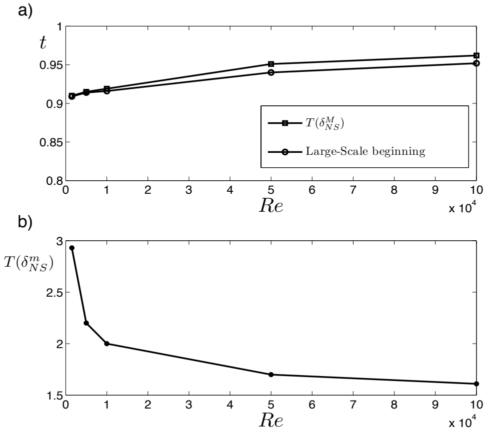

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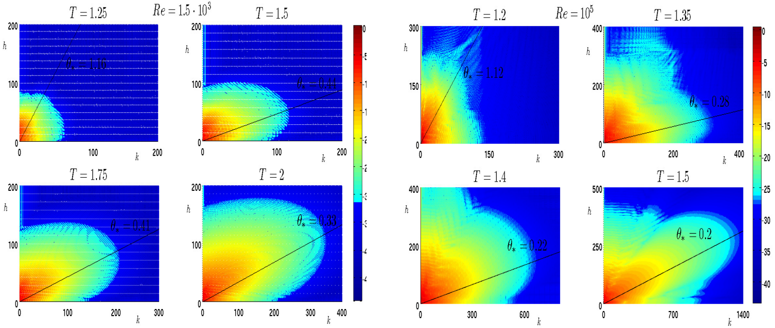
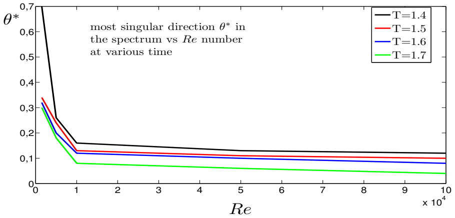




Complex Singularity Tracking

The “singularity tracking method” gives the singularity closest to the real plane. We analyze the full set of complex singularities considering the wall shear for Prandtl and NS.

Padé approximants

- This methodology allows to determine the position of the complex singularities.
- Does not provide clear information regarding the characterization of complex singularities.

Pólya method

- This methodology allows to capture the position and the characterization of “ branches ” or poles of a power series.
- When two or more singularities are close together, there are numerical problems.

This method was introduced by Pauls & Frish *J. Stat. Phys. 07'*.

Complex Singularity Tracking

Padé approximants: given a complex function as a Taylor (Fourier) series

$$u(z) = \sum_{k=0}^{\infty} u_k z^k,$$

the Padé approximants $P_{L/M}$ is the rational function which approximates u:

$$P_{L/M} := \frac{\sum_{i=0}^{L} a_i z^i}{1 + \sum_{j=1}^{M} b_j z^j} = u(z) + O\left(z^{L+M+1}\right),$$

The a_i, b_j are determinate from the following linear system $(ill - posed)$:

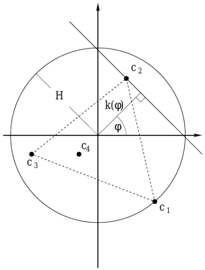
$$\sum_{i=0}^{\min(\alpha,M)} b_i c_{\alpha-i} = a_\alpha \quad \alpha = 0, \ldots, L; \qquad \sum_{i=0}^{M} b_i c_{L+\beta-i} = 0, \quad \beta = 1, \ldots, M.$$

This method only provides information on the location of the various complex singularity, even those outside the radius of convergence, but does not provide information on their characterization.

Complex Singularity Tracking

Pólya method: Let f be an analytic function:

$$\underbrace{f(z) = \sum_{k=0}^{N} a_k / z^{k+1}}_{Taylor(Fourier)}.$$



Let H be the radius of convergence, and K the the smallest convex and compact set that contains the singularities. Then:

- F define an entire function of exponential type,

$$\underbrace{F(\zeta) = \sum_{k=0}^{N} a_k \zeta^k / n!}_{Borel-Laplace \quad Transform}$$

- the following relation holds $k(\varphi) = h(-\varphi)$ with

$$\begin{aligned} h(\varphi) &= \limsup_{r\to\infty} \ln(F(re^{i\varphi}) \quad \text{indicatrix} \quad \text{function} \\ k(\varphi) &= \sup_{Z\in K} \Re(Ze^{-i\varphi}) \quad \text{supporting} \quad \text{function,} \end{aligned}$$

- $H = \sup_\varphi h(\varphi)$.

Complex Singularity Tracking

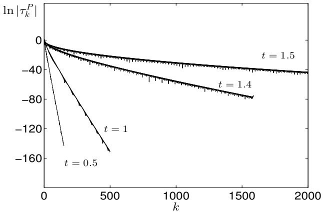
Consider $f(z) \sim \sum_{j=1}^{n} (z - c_j)^{\alpha_j - 1}$, with n complex singularities in $c_j = |c_j| e^{-i\gamma_j}$, then the asymptotic behaviour along direction $re^{i\varphi}$, with $\varphi_{j-1} < -\varphi < \varphi_j$:

$$\begin{aligned} \left|F(re^{i\varphi})\right| &= C r^{\alpha_j} e^{h(\varphi) r} \left[1 + \varepsilon(r)\right], \\ h(\varphi) &= |c_j| \cos(\varphi - \gamma_j). \end{aligned}$$

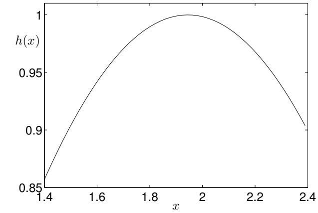
- From the *indicatrix function* $h(\varphi)$ (which is a piecewise-cosine function) one obtains informations of the complex singularity location.
- Form $\alpha_j(\varphi)$ one obtains the algebraic character of the complex singularity.
- High-precision numerical computation is used if two or more singularities are close together.

Both the method of Padé that Pólya, if applied individually, do not give a complete picture of the complex singularity of a given function, therefore they should be used together to provide a solid framework for analysing complex singularity.

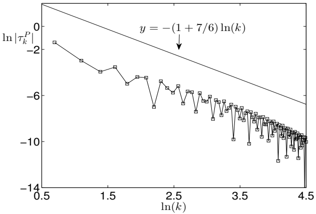
Prandtl Singularity



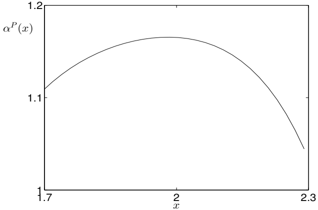
(d) The Fourier spectrum of τ_w^P.



(e) The indicatrix function $h(x)$ at $t_s = 1.5$ which is a cosine centred in $x_s \approx 1.94$.

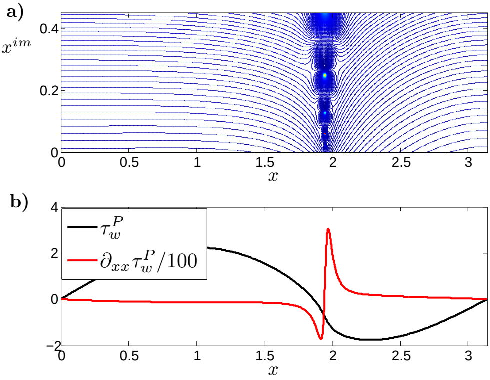


(f) The algebraic type of singularity of τ_w^P.



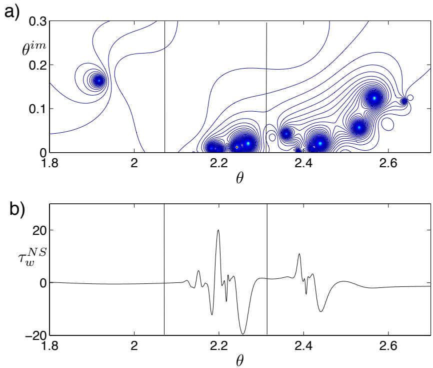
(g) The algebraic decay rate is $\alpha^P \approx 7/6$.

Prandtl Singularity



- The Padè approximant $P_{200/200}$ of τ_w^P at $t = 1.5$.
- τ_w^P at $t = 1.5$ shows a blow up in its second derivative.

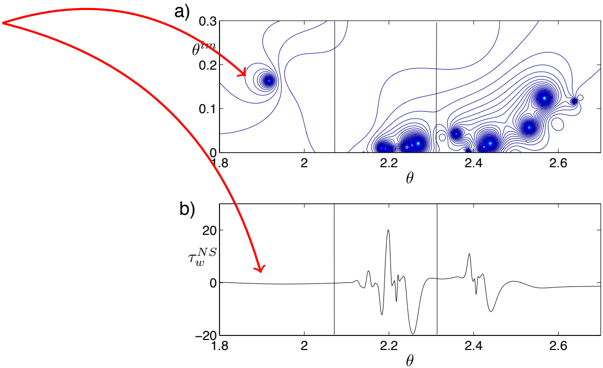
NS Singularity



- The Padè approximat $P_{300/300}$ of τ_w^{NS} for $Re = 10^5$ at $t = 1.58$. Are distinguishable three distinct groups of singularities.

NS Singularity

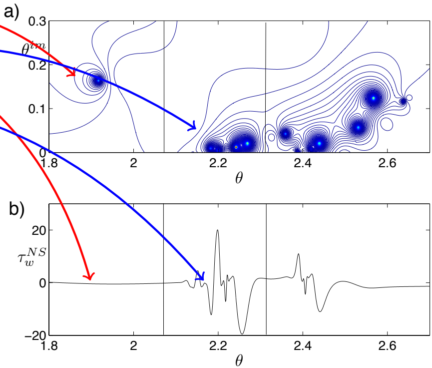
- Prandtl syngularity (s_P)



- The Padè approximat $P_{300/300}$ of τ_w^{NS} for $Re = 10^5$ at $t = 1.58$. Are distinguishable three distinct groups of singularities.

NS Singularity

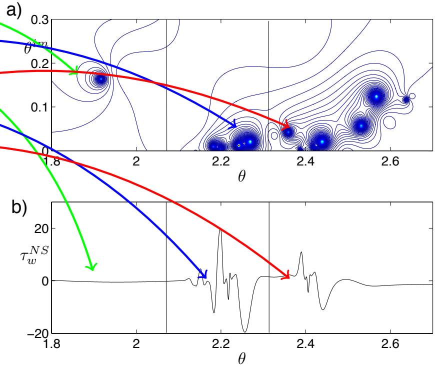
- Prandtl singularity (s_P)
- Large–scale singularity (s_{ls})-related to the formation of the Large–Scale interaction.



- The Padè approximat $P_{300/300}$ of τ_w^{NS} for $Re = 10^5$ at $t = 1.58$. Are distinguishable three distinct groups of singularities.

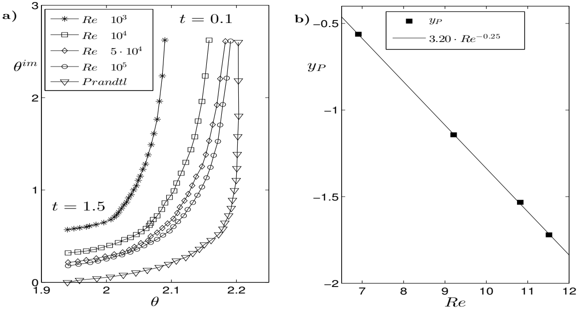
NS Singularity

- Prandtl singularity (s_P)
- Large–scale singularity (s_{ls})-related to the formation of the Large–Scale interaction.
- Small–scale singularity (s_{ss})-related to the formation of the Small–Scale interaction.

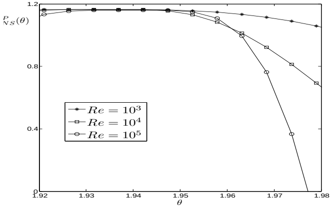


- The Padè approximat $P_{300/300}$ of τ_w^{NS} for $Re = 10^5$ at $t = 1.58$. Are distinguishable three distinct groups of singularities.

NS: the s_P singularity

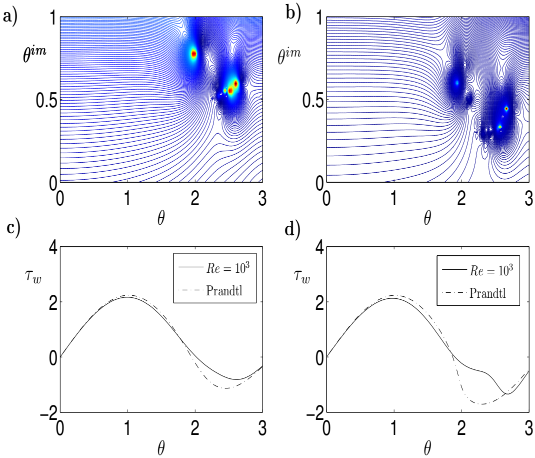


The evolution in time of s_P and its minimum distance from the real axis y^P respect to Re.



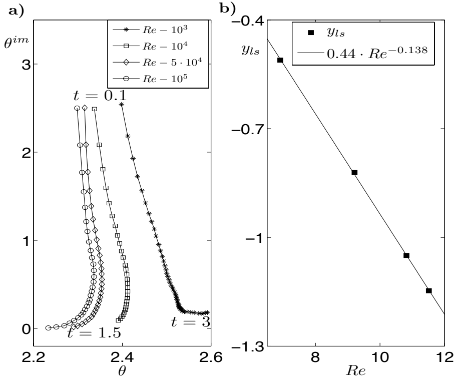
The algebraic characterization of s_P at $t = 1.5$.

NS: the s_{ls} singularity



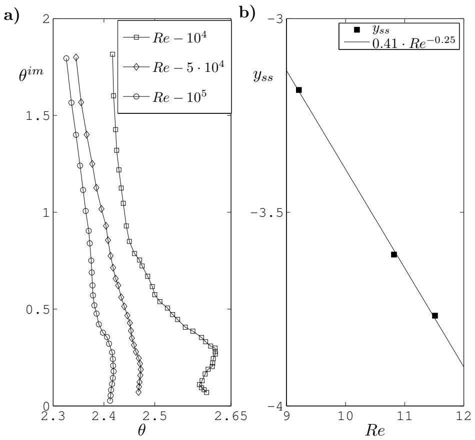
The comparison between τ_w of NS (with $Re = 10^3$) and Prandtl, and the Padè approximant $P_{200/200}$ at $t = 1$ (a, b) and $t = 1.45$ (b, d). At $t = 1.45$, s_{ls} is related to the formation of a gradient in τ_w close to its minimum.

NS: the s_{ls} singularity

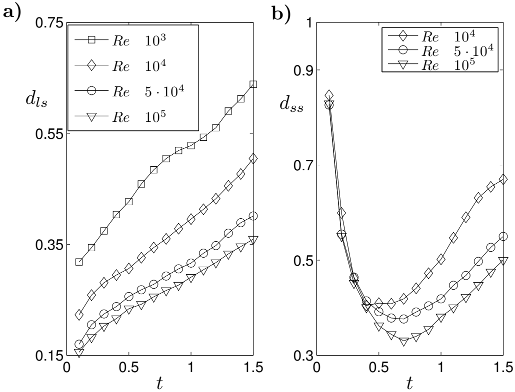


- The evolution in time in the complex plane (θ, θ^{im}) of s_{ls}. After the large-scale interaction, the real part of s_{ls} moves "upstream" along the cylinder for $Re = 10^4 - 10^5$, while the opposite occurs for $Re = 10^3$.
- At T_{LS} the time when the large-scale interaction starts, the singularity remains at a distance y_{ls} from the real axis, which depends on Re as $0.44 \cdot Re^{-0.138}$ ($y_{ls} \to 0$ for $Re \to \infty$).
- The singularities of τ_w^{NS} have $\alpha_{NS}^{s_{ls}} = 1/2$ $(t = 1)$.

NS: the s_{ss} singularity



- The evolution in time in the complex plane (θ, θ^{im}) of s_{ss}. When the small-scale interaction starts the singularity is a distance y_{ss} from the real axis, which depends on Re as $0.41 \cdot Re^{-0.25}$ ($y_{ss} \to 0$ as $Re \to \infty$).
- The singularities have $\alpha = 1/2$.



- The evolution in time of the distance d_{ls} in the complex plane between s_P and s_{ls} for different Re, and the evolution in time of the distance d_{ss} between s_P and s_{ss}.

Conclusions:

In both cases, the distance decreases with the increase of Re, and this supports the conjecture that asymptotically all singularities are reduced to the singularity of Prandtl s_P.

Navier BC:

- Navier Boundary Condition:

$$\mathbf{u}^{NS} \cdot \nu = 0, \qquad \frac{1}{Re}\left(\frac{\partial \mathbf{u}^{NS}}{\partial \nu} + C\mathbf{u}^{NS}\right) \cdot \tau = -\beta(Re)\mathbf{u}^{NS} \cdot \tau \quad .$$

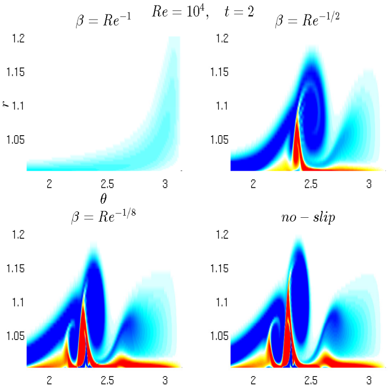
  - If

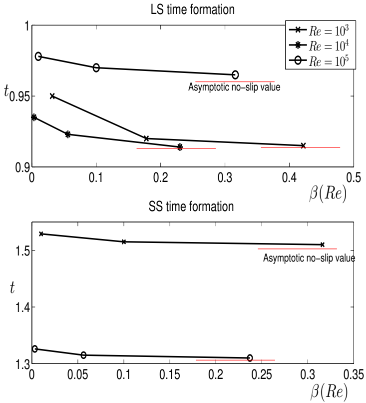
$$\lim_{Re \to +\infty} \frac{1}{Re}\left(\frac{\partial \mathbf{u}^{NS}}{\partial \nu}\right) = 0,$$

    then each weak limit $\mathbf{u}^{NS}$ is a dissipative solution of the Euler equations.
  - Dirichlet BC: $\beta = \infty$

In recent years several of the classical fluid problems of this type have been recast to model flows on a nanoscale or microscale and the Navier-slip conditions become relevant in certain applications related to hemodynamics and high-altitude flows as well.
In the present study Navier boundary conditions allows for slip on the disk, and we consider $\beta(Re) = Re^{-\alpha}$.

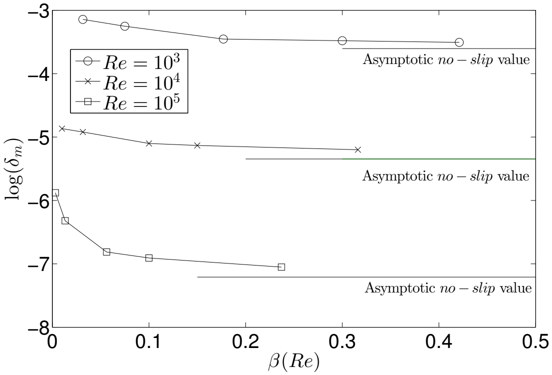
Navier BC:





- The value $\beta = Re^{-1}$ is somewhat critical, as for $\beta = Re^{-\alpha}$, $\alpha < 1$ both large–scale and small–scale interactions forms as in the no-slip case.
- For $\beta = Re^{-1}$ no reciruclation region forms and viscous-inviscid no interactions are present.
- The time formations of both large and small scale interaction are delayed, and as $\alpha \to 0$ this time tends to the time determined by the no-slip condition.

Complex singularity analysis for Navier BC



For $\beta = Re^{-1}$ no complex singularities are detected, while for $\beta = Re^{-\alpha}$, $\alpha < 1$, NS solutions have complex singularities. As $\alpha \to 0$ The width of analicity of NS solution tends to that predicted by the slip-case.

Bibliography:

- F. Gargano, M. Sammartino, V. Sciacca,K. Cassel, *Analysis of Complex Singularity in High-Reynolds-numbers Navier-Stokes solutions*, **J. Fluid Mech,**, vol.747 (2014).
- F. Gargano, M. Sammartino, V. Sciacca,K. Cassel, *Viscous-Inviscid Interactions in a Boundary-Layer Flow Induced by a Vortex Array*, **Acta Applic. Math.**, vol. 132 (2014).
- F. Gargano, M. Sammartino, V. Sciacca,High Reynolds number Navier-Stokes solutions and boundary layer separation induced by a rectilinear vortex, **Computer and Fluids**, vol.52 (2011).
- F. Gargano, A.M. Greco, M. Sammartino, V. Sciacca, Unsteady Separation for High Reynolds numbers flow, **Proc. WASCOM 2009**, (2010).
- F. Gargano, M. Sammartino, V. Sciacca, Singularity Formation and separation Phenomena in Boundary Layer Theory, **Physica D**, vol.238(19) (2009).
- F. Gargano, M.C Lombardo, M. Sammartino, V. Sciacca, Singularity and separation in Boundary Layer, **London Math. Society Lecture Notes Series**, (2009).
- G. Della Rocca, F. Gargano, M. Sammartino, V. Sciacca, High Reynolds number Navier-Stokes solutions and boundary layer separation induced by a rectilinear vortex array, **Proc. WASCOM 2007**, (2008).
- G. Della Rocca, M.C. Lomabrdo, M. Sammartino, V. Sciacca, Singularity Tracking for Camassa-holm and Prandtl's equations, **App.Numer.Maths**, vol.56(8) (2006).