Domain Decomposition methods for Isogeometric Analysis and applications to computational electrocardiology

Simone Scacchi	University of Milan

Joint work with

Lourenço Beirão da Veiga	University of Milan
Lara A. Charawi	University of Pavia
Durkbin Cho	Dongguk University
Luca F. Pavarino	University of Milan
Olof Widlund	Courant Institute
Stefano Zampini	KAUST

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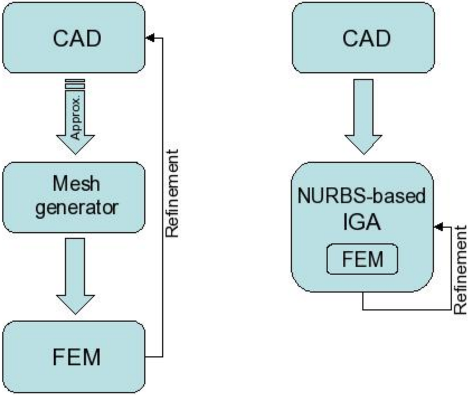
Isogeometric Analysis

Isogeometric analysis (IGA) [Hughes, Cottrell, Bazilevs 2005] uses NURBS spaces (the same spaces used in CAD) as discrete spaces for the problem approximation (Galerkin, Collocation, etc..).



Isogeometric Analysis

Isogeometric analysis (IGA) [Hughes, Cottrell, Bazilevs 2005] uses NURBS spaces (the same spaces used in CAD) as discrete spaces for the problem approximation (Galerkin, Collocation, etc..).

This leads to a series of advantages, including

- exact geometry representation;
- easier refinement of mesh and spaces;
- easy to handle spaces that are highly regular (C^1, C^2, etc..) across mesh edges
  - better efficiency in approximation
  - application to higher order problems
  - computation of derived quantities (normals, strains, etc...)
  - eigenvalues, ...

B-splines in one dimension

A space of univariate B-splines on the interval $[a, b]$ is uniquely defined by a polynomial degree p and an (open) knot vector ξ

$$a = \xi_0 = \ldots = \xi_p < \xi_{p+1} \leq \xi_{p+2} \leq ... \leq \xi_{n-1} < \xi_n = \ldots = \xi_{n+p} = b$$

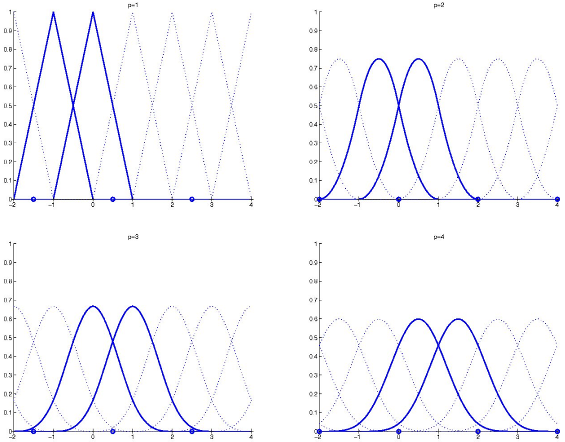
as the span of the basis functions

$$S_h = \operatorname{span}\left\{ N_i^p \ : \ i = 1, 2, ..., n \right\}.$$

The basis functions N_i^p, that depend on ξ and p can be defined for instance by an iterative formula.

Univariate B-splines

Example of B-spline basis functions in the periodic case, $p = 1, 2, 3, 4$ (no knot repetitions):



Multivariate B-splines and NURBS

B-spline spaces in higher dimensions are built with a tensor product construction. For instance for $d = 2$ the basis functions are $(1 \leq i \leq n, 1 \leq j \leq m)$

$$N_{i,j}^{p,q}(\xi, \eta) = N_i^p(\xi) N_j^q(\eta) \qquad \forall (\xi, \eta) \in [0, 1]^2,$$

where the one-dimensional basis functions may be based on different knot vectors $\boldsymbol{\xi}, \boldsymbol{\eta}$ and polynomial degrees p, q.

The B-Spline space is defined as the span

$$S_h = \text{span}\left\{ N_{i,j}^{p,q} \; : \; 1 \leq i \leq n, 1 \leq j \leq m \right\}.$$

$$\widehat{\Omega} = [0, 1]^2 \quad \text{is} \quad \text{the parametric domain}.$$

Multivariate B-splines and NURBS

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where the one-dimensional basis functions may be based on different knot vectors $\boldsymbol{\xi}, \boldsymbol{\eta}$ and polynomial degrees p, q.

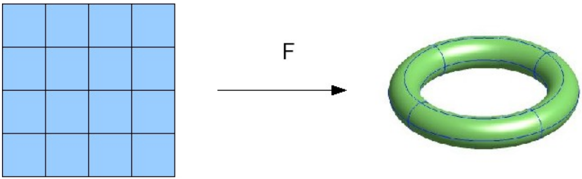
NURBS spaces and basis functions (in parametric domain) are defined by

$$N_h = \operatorname{span}\left\{ R_{i,j}^{p,q} \ : \ 1 \le i \le n, 1 \le j \le m \right\}, \quad R_{i,j}^{p,q} = \frac{N_{i,j}^{p,q}}{w},$$

with $w \in S_h$ a positive weight function fixed once and for all.

Geometry and mapped NURBS spaces

The domain of interest Ω is the image of a NURBS map $\mathbf{F}$.



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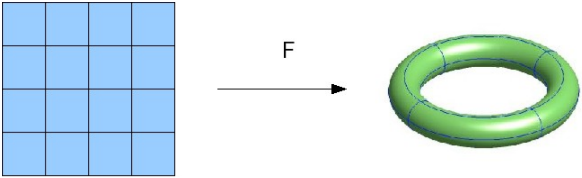

The NURBS space in physical space is simply the push forward

$$V_h = \left\{ v_h \circ \mathbf{F}^{-1} \ : \ v_h \in N_h \right\}.$$

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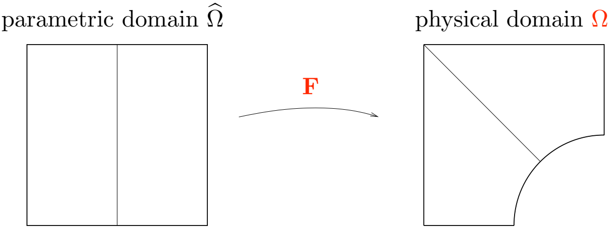

Isoparametric paradigm: The space N_h (and thus V_h) is obtained by $h - p - k$ refinement of the initial coarse space used to define $\mathbf{F}$ (and w).

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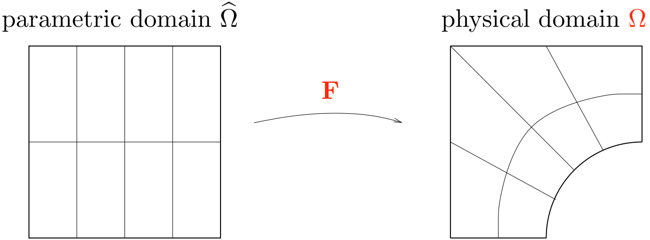

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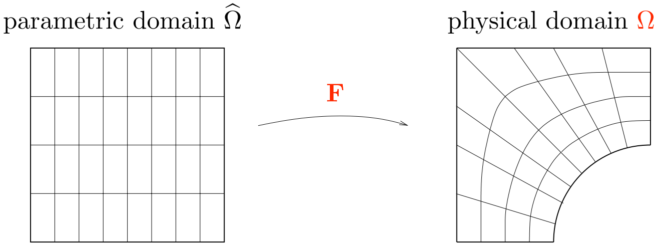

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Approximation properties of mapped NURBS

It exists a quasi-interpolant $\Pi_h : L^2(\Omega) \to V_h$ such that

Theorem

It exists $C = C(p) \in \mathbb{R}$ such that for all K elements of the physical mesh

$$|f - \Pi_h f|_{H^m(K)} \leq C \, (h_K)^{s-m} \, |f|_{H^s(\widetilde{K})} \qquad \forall f \in H^s(\Omega),$$

where $\widetilde{K}$ is an extended patch and $0 \leq m \leq s \leq p + 1$.

- the proof can be found in [Bazilevs, Beirão da Veiga, Cottrell, Hughes, Sangalli, 2006]
- an anisotropic version, obtained with different techniques, can be found in [Beirão da Veiga, Cho, Sangalli, 2011]
- under additional assumptions, full hpk estimates are derived in [Beirão da Veiga, Buffa, Rivas, Sangalli, 2010]

IGA and preconditioners

Clearly, the condition number of IGA problems grows (as for FEM) when the space is enriched (in p or h).
Some references for IGA solvers:

- N. Collier, D. Pardo, L. Dalcin, M. Paszynski and V.M. Calo. *The cost of continuity: a study of the performance of isogeometric finite elements using direct solvers.* CMAME 2012.
- L. Beirão da Veiga, D. Cho, L. F. Pavarino, S. Scacchi, *Overlapping Schwarz methods for Isogeometric Analysis*. SINUM 2012.
- S. Kleiss, C. Pechstein, B. Juttler, S. Tomar, *IETI - Isogeometric Tearing and Interconnecting.* CMAME 2012.
- L. Beirão da Veiga, D. Cho, L.F. Pavarino, S. Scacchi, *BDDC preconditioners for Isogeometric Analysis*. M3AS 2013.
- A. Buffa, H. Harbrecht, A. Kunoth, G. Sangalli, *BPX-preconditioning for isogeometric analysis*. CMAME 2013.
- ...

Overlapping Additive Schwarz preconditioner: the subdomains partition

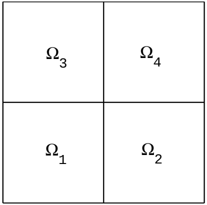
We consider the model problem

$$\begin{cases} -\operatorname{div}(\rho \nabla u) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

which, after discretization, reduces to the variational problem

$$\text{find } u_h \in V_h : \quad a(u_h, v) = (f, v) \quad \forall v \in V_h.$$

The first step is to divide the parametric space into N non-overlapping subdomains Ω_i, e.g. [Toselli-Widlund, 2004].



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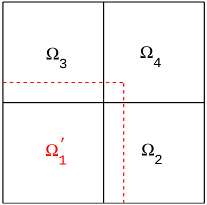
Then extend each subdomain to obtain a partition of the parametric space into N overlapping subdomains Ω_i'



Overlapping Additive Schwarz preconditioner: the operator construction

- Introduce the local NURBS spaces related to subdomains

$$V_i := \{\mathbf{v} \in V_h \, : \, \mathbf{v}(x) = 0 \ x \in \Omega \backslash \Omega_i^{'}\}, \quad i = 1, ..., N$$

- Introduce the coarse NURBS space

$$V_0 \subset V_h$$

- Define the projections $\mathbf{T}_i : V_h \to V_i$, $i = 0, ..., N$

$$a(\mathbf{T}_i \mathbf{u}, \mathbf{v}) = a(\mathbf{u}, \mathbf{v}) \quad \forall \mathbf{v} \in V_i.$$

- The two-level Additive Schwarz operator is given by

$$\mathbf{T}_{OAS} = \mathbf{T}_0 + \mathbf{T}_1 + ... + \mathbf{T}_N = P_{OAS}^{-1} A.$$

where P_{OAS}^{-1} is the Additive Schwarz preconditioner and A the original stiffness matrix.

Overlapping Additive Schwarz preconditioner: convergence rate bound

Theorem

The condition number of the 2-level additive Schwarz preconditioned isogeometric operator $\mathbf{T}_{OAS}$ *is bounded by*

$$\kappa_2(\mathbf{T}_{OAS}) \leq C \left(1 + \frac{H}{\gamma}\right),$$

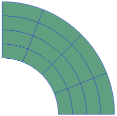
where $\gamma = \gamma(h)$ *is the overlap parameter and* C *is a constant independent of* h, H, N, γ *(but not of degree* p *and regularity* k*).*

More details and proof in:

L. Beirão da Veiga, D. Cho, L. F. Pavarino, S. Scacchi. *Overlapping Schwarz methods for Isogeometric Analysis*. SIAM J. Numer. Anal. 2012

2D tests: OAS scalability in N and optimality in H/h

Quarter of Ring domain



NURBS parameters $p = 3, k = 2$
2-lev OAS preconditioner with $\gamma = 2h$

Condition number $\kappa_2(T_{OAS})$ and iteration counts it. as a function of the number of subdomains N and mesh size inverse $1/h$:

	$1/h = 8$		$1/h = 16$		$1/h = 32$		$1/h = 64$		$1/h = 128$	
N	κ_2	it.	κ_2	it.	κ_2	it.	κ_2	it.	κ_2	it.
2×2	7.30	14	6.98	14	11.44	17	20.58	22	38.97	30
4×4			8.12	18	10.62	20	19.60	23	37.72	32
8×8					8.41	19	13.92	21	29.88	27
16×16							8.32	19	15.50	22
32×32									8.34	19

Linear Elasticity, 3D tests: OAS scalability in N

3D cubic domain

NURBS parameters $p = 3, k = 2$
2-lev OAS preconditioner with fixed ratio $H/h = 4$

Young modulus $E = 6e + 6$, Poisson ratio $\nu = 0.3$

N	$\gamma = 2h$		$\gamma = 4h$	
	$\kappa_2 = \lambda_{max}/\lambda_{min}$	it.	$\kappa_2 = \lambda_{max}/\lambda_{min}$	it.
$2 \times 2 \times 2$	$17.16 = 8.03/0.47$	23	$9.27 = 8.25/0.89$	21
$3 \times 3 \times 3$	$22.84 = 8.04/0.35$	28	$12.80 = 9.68/0.76$	25
$4 \times 4 \times 4$	$20.06 = 8.04/0.40$	27	$12.01 = 9.47/0.79$	24
$5 \times 5 \times 5$	$20.52 = 8.04/0.39$	27	$12.37 = 9.53/0.77$	25
$6 \times 6 \times 6$	$20.62 = 8.05/0.39$	27	$12.51 = 9.56/0.76$	25

The Bidomain model of cardiac tissue

Reaction-Diffusion system coupled with an ODEs system.

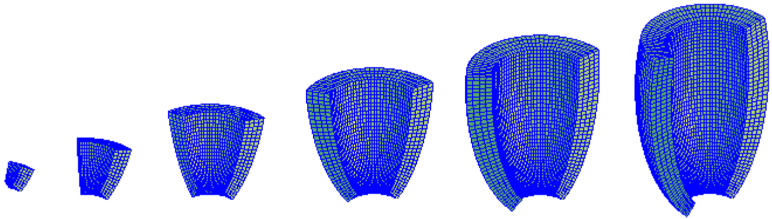
- Given $I_{app}^{i,e}$ (applied currents per unit volume),
- Find v, u_e and w (gating variables), such that

$$\begin{cases} \chi C_m \frac{\partial v}{\partial t} - \mathrm{div}(D_i \nabla (v + u_e)) + \chi I_{ion}(v, w) & = I_{app}^i \\ -\mathrm{div}((D_i + D_e)\nabla u_e) - \mathrm{div}(D_i \nabla v) & = I_{app}^e + I_{app}^i \\ \frac{\partial w}{\partial t} - R(v, w) & = 0 \end{cases}$$

+ 0 Neumann b. c. and initial conditions for v, w.
+ compatibility conditions.
$D_{i,e}$ = conductivity tensors, χ =ratio of membrane area/tissue volume; C_m =surface capacitance; I_{ion} =ionic current resulting from the membrane model R.

(see Pennacchio, Savaré, Colli Franzone. SIAM J. Math. Anal. 2006)

Bidomain model: Scalability test



N	Unprec. it.	Unprec. κ_2	1-level OAS it.	1-level OAS κ_2	2-level OAS it.	2-level OAS κ_2
$2 \times 2 \times 1$	765	2.85e4	14	10.34	11	5.72
$4 \times 4 \times 1$	1236	4.92e4	27	58.61	10	6.62
$6 \times 6 \times 1$	1539	7.30e4	35	1.42e2	9	6.27
$8 \times 8 \times 1$	1949	1.01e5	47	2.66e2	8	5.53
$10 \times 10 \times 1$	2180	1.14e5	55	4.52e2	8	5.50
$12 \times 12 \times 1$	2307	1.25e5	63	6.67e2	8	5.50

L. A. Charawi. Isogeometric Overlapping Additive Schwarz Preconditioners in Computational Electrocardiology. PhD Thesis, University of Milan, 2014

A BDDC preconditioner

We will now present a Balancing Domain Decomposition by Constraints (BDDC) preconditioner for Isogeometric Analysis of elliptic problems in primal form (standard diffusion model problem).

- BDDC was introduced in [Dohrmann, 2003] and analyzed first in [Mandel, Dohrmann, 2003]
- the results presented in this talk can be found in [Beirão da Veiga, Cho, Pavarino, S., M3AS, 2013]; [Beirão da Veiga, Pavarino, S., Widlund, Zampini, SISC, 2014].

Schur complement system

We consider the model problem

$$\begin{cases} -\operatorname{div}(\rho \nabla u) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

As usual, the first step is to divide the parametric space into (rectangular) subdomains, e.g. [Toselli-Widlund, 2004].

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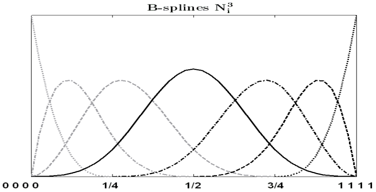
In the case of IGA, the higher continuity (and thus the larger support) of basis functions means that in general one cannot reduce the problem to the skeleton.



Schur complement system

The concept of a “fat boundary” (that is easily understood in terms of degrees of freedom) must be introduced.

Schur complement system

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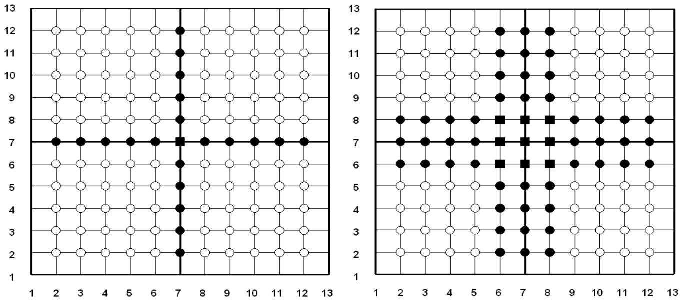
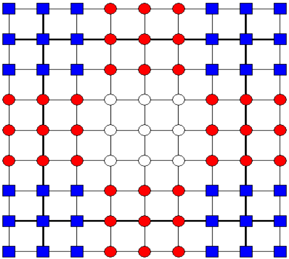
Example (in index space, dim.$=2$):



Primal (coarse) degrees of freedom (dim.= 2)

The coarse space degrees of freedom are associated to “fat corners”:



Primal dof (blue in figure)
Dual dof (red in figure)
Interior condensed dof (white in figure)

Choice of scaling operators

We propose and analyze three possible choices:

- "Standard" ρ scaling:

$$\delta_{ij}^{(k)^\dagger} = \rho_k / \Big(\sum_{\ell \in \mathcal{N}_{ij}} \rho_\ell \Big).$$

- Stiffness scaling (balances energy of basis functions):

$$\delta_{ij}^{(k)^\dagger} = s_k(N_{i,j}^{p,q}, N_{i,j}^{p,q}) / \Big(\sum_{\ell \in \mathcal{N}_{ij}} s_\ell(N_{i,j}^{p,q}, N_{i,j}^{p,q}) \Big).$$

- Deluxe scaling (balances the local Schur complements), first introduced in Dohrmann and Widlund 2013

Theoretical condition number bounds

Theorem

The condition number of the BDDC preconditioned isogeometric operator is bounded by

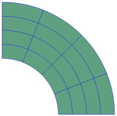
$$\kappa_2(P) \le C\left(1 + \log^2(H/h)\right) \qquad \rho \textit{ and deluxe scaling},$$

$$\kappa_2(P') \le C\left(1 + \log\left(\frac{H}{h}\right)\right)\frac{H}{h} \qquad \textit{stiffness scaling},$$

where the constant C is independent of H (subdomain size), h (fine mesh size).

2D tests: BDDC quasi-optimality

Quarter of Ring domain (2D):

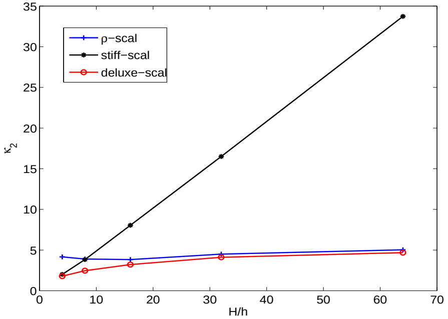


NURBS parameters $p = 2, k = 1$
BDDC preconditioner with $N = 4 \times 4$ subdomains

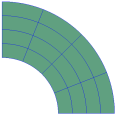
H/h	ρ-scal. κ_2	it.	stiff.-scal. κ_2	it.	deluxe-scal. κ_2	it.
4	4.16	14	2.01	9	1.79	8
8	3.90	14	3.83	13	2.46	9
16	3.83	14	8.05	16	3.22	10
32	4.50	14	16.50	21	4.11	12
48	5.03	15	33.73	25	4.68	12

2D tests: BDDC quasi-optimality

Condition numbers κ_2 of the BDDC preconditioned system with respect to the ratio $\frac{H}{h}$



2D tests: BDDC scalability



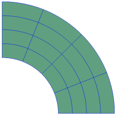
Quarter of Ring domain (2D):

NURBS parameters $p = 2, k = 1$
BDDC preconditioner with fixed ratio $H/h = 4$

N	ρ-scal. κ_2	ρ-scal. it.	stiff.-scal. κ_2	stiff.-scal. it.	deluxe-scal. κ_2	deluxe-scal. it.
2×2	3.72	12	1.65	8	1.17	5
4×4	4.16	14	2.01	9	1.79	8
8×8	4.20	14	2.27	10	2.11	9
16×16	4.07	14	2.41	10	2.30	10
32×32	3.97	13	2.50	11	2.40	10

2D tests: BDDC behavior for high p, k

Quarter of Ring domain (2D):

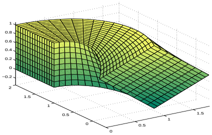


Maximal spline regularity $k = p - 1$
BDDC preconditioner with fixed ratio $H/h = 16$ and $N = 4 \times 4$ subdomains

	ρ-scal.		stiff.-scal.		deluxe-scal.	
p	κ_2	it.	κ_2	it.	κ_2	it.
2	3.83	14	8.15	16	3.22	10
3	76.52	53	15.05	20	2.68	10
4	2838.56	141	11.09	22	2.41	9
5	147769.26	548	31.62	35	2.19	9
6			84.75	71	2.04	9
7			333.84	113	1.91	8
8			1031.59	229	1.80	8
9			3830.01	388	1.72	8
10			12761.38	807	1.62	9

3D tests: BDDC scalability

NURBS parameters $p = 3, k = 2$
BDDC preconditioner with fixed ratio $H/h = 6$



N	stiff.-scal. κ_2	stiff.-scal. it.	deluxe-scal. κ_2	deluxe-scal. it.
$2 \times 2 \times 2$	8.94	24	1.67	9
$3 \times 3 \times 3$	9.21	27	1.81	10
$4 \times 4 \times 4$	9.27	28	1.85	10
$5 \times 5 \times 5$	9.35	28	1.86	10
$6 \times 6 \times 6$	9.38	29	1.92	10

Conclusions

- Isogeometric analysis is a fast growing recent (2005) technology for the numerical approximation of PDEs
- Preconditioners and solvers in IGA are needed for large scale problems
- We have presented OAS and BDDC preconditioners for IGA, together with theoretical results on scalability and quasi-optimality
- 2D and 3D numerical results have validated the theoretical estimates