Motion of aerobic bacteria in liquids

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Flagellated aerobic bacteria in liquids

- *Bacillus subtilis in water*
- *Pseudomonas aeruginosa in blood*
- *Legionella pneumophila in Vila Franca de Xira, Portugal (November 2014)*

Flagellated aerobic bacteria in liquids

- *oxytaxis*
- *metabolism*
- *cell–cell signaling*
- *buoyancy*
- *diffusion*
- *mixing*
- *proliferation/death*

PDE model

$$\partial_t n + u \cdot \nabla n - \Delta(n^m) = -\nabla \cdot (\chi(c) n \nabla c) + f(n), \qquad (1)$$

$$\partial_t c + u \cdot \nabla c - \Delta c = -k(c) n, \qquad (2)$$

$$\partial_t u + u \cdot \nabla u - \Delta u + \nabla p = -n \nabla \phi, \qquad (3)$$

$$\nabla \cdot u = 0, \qquad (4)$$

$$\frac{\partial n^m(t,x)}{\partial \nu} = 0, \ \frac{\partial c(t,x)}{\partial \nu} = 0, \ u(t,x) = 0, \ x \in \partial\Omega, \qquad (5)$$

$$n(0,x) = n_0(x), \ c(0,x) = c_0(x), \ u(0,x) = u_0(x), \ x \in \Omega. \qquad (6)$$

Involved quantities

$T > 0$,
$\Omega \subset \mathbb{R}^d$, with $d = 2, 3$, a bounded domain or the whole space $\mathbb{R}^d$ itself,
$Q_T = (0,\ T) \times \Omega$,
$c(t,\ x) : Q_T \to \mathbb{R}$, $n(t,\ x) : Q_T \to \mathbb{R}$ are the oxygen and cell concentrations, resp.,
$u(t,\ x) : Q_T \to \mathbb{R}^d$ is the fluid velocity, $p(t, x) : Q_T \to \mathbb{R}$ is the hydrostatic pressure,
The scalar functions k, χ and f determine the oxygen consumption rate, chemotactic sensitivity, and bacterial growth, resp.,
$\phi : Q_T \to \mathbb{R}$ is the physical potential,
$m \geq 1$ is the nonlinear diffusion exponent.
The cases $m = 1$ and $f \equiv 0$ are not excluded.

Previous results ($m = 1$, full Navier-Stokes)

- Existence of local weak solutions.
- 2D: Under some more or less restrictive assumptions on k and χ, and on the domain $\Omega \subset \mathbb{R}^2$ (bounded and convex/whole plane), one can prove existence of global regular/weak solutions.
- 3D: Global regular/weak solutions exist for $\Omega = \mathbb{R}^3$ when the initial datum is a small smooth perturbation of the steady state $(n_0 = const, 0, 0)$, or when $k/\chi = const$, or when $\chi = const$ and k is a linear function.

(see Winkler 2012, Winkler 2014, Chae et al. 2011, Lorz 2010, Di Francesco et al. 2010)

The supercritical case

Let $m > \frac{d+1}{3}$. Let $\phi \in L_1(0,T;L_{1,loc})$ with $\nabla\phi \in L_2(0,T;L_\infty)$. Let k, χ and f be continuously differentiable functions, $\chi' \geq 0$, $k \geq 0$, $k(0) = 0$, $f(0) \geq 0$ (but $f(0) = 0$ for $\Omega = \mathbb{R}^d$) and

$$f(y) \leq f(0) + Cy \tag{7}$$

for $y \geq 0$.

Let $n_0 \in L_1 \cap L_{\max(1,m/2)}$, $n_0 \ln n_0 \in L_1$, $\langle\cdot\rangle n_0(\cdot) \in L_1$, $c_0 \in H^1 \cap L_\infty$, $n_0 \geq 0$, $c_0 \geq 0$, $u_0 \in H$. Then problem (1)–(6) possesses a nonnegative [1] weak solution (c,n,u) [2] [3].

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[1] i.e. $c, n \geq 0$

[2] roughly speaking, in a Leray-Hopf sense

[3] Here, $\langle x\rangle = \sqrt{1+|x|^2}$ for $\Omega = \mathbb{R}^d$, and $\langle x\rangle = 1$ for bounded Ω

The subcritical case

Let $1 \leq m \leq \frac{d+1}{3}$. Suppose that

$$f(y) + C_f y^2 \leq f(0) + Cy \tag{8}$$

with some positive C_f independent of $y \geq 0$, and the remaining assumptions of the previous page hold. Then problem (1)–(6) possesses a nonnegative weak solution.

Moreover, if $\Omega = \mathbb{R}^2$, $m = 1$, f, χ and k are C^3-smooth, $f'(y) + |f''(y)| \leq C$ for $y \geq 0$, $\nabla\phi \in W^2_\infty$ (and independent of t), $n_0 \in H^2$, $c_0 \in H^3$, $u_0 \in H^3$, there exists a unique nonnegative classical solution to (1)–(6).

Attractors without uniqueness

Basic framework (V. and Zvyagin, 2008):
Let E and E_0 be Banach spaces, $E \subset E_0$, E is reflexive. Fix some set

$$\mathcal{H}^+ \subset C([0,+\infty); E_0) \cap L_\infty(0,+\infty; E)$$

of solutions (strong, weak, etc.) for any given autonomous differential equation or boundary value problem. Hereafter, the set $\mathcal{H}^+$ will be called the *trajectory space* and its elements will be called *trajectories*. Generally speaking, the nature of $\mathcal{H}^+$ may be different from the just described one.
Let $T(h)$ be the translation (shift) operator,

$$T(h)(u)(t) = u(t+h).$$

Attracting and absorbing sets

A set $P \subset C([0, +\infty); E_0) \cap L_\infty(0, +\infty; E)$ is called *attracting* (for the trajectory space $\mathcal{H}^+$) if for any set $B \subset \mathcal{H}^+$ which is bounded in $L_\infty(0, +\infty; E)$, one has

$$\sup_{u \in B} \inf_{v \in P} \|T(h)u - v\|_{C([0,+\infty);E_0)} \underset{h \to \infty}{\to} 0.$$

A set $P \subset C([0, +\infty); E_0) \cap L_\infty(0, +\infty; E)$ is called *absorbing* (for the trajectory space $\mathcal{H}^+$) if for any set $B \subset \mathcal{H}^+$ which is bounded in $L_\infty(0, +\infty; E)$, there is $h \geq 0$ such that $T(t)B \subset P$ for all $t \geq h$.

A set $\mathcal{U} \subset C([0,+\infty);E_0) \cap L_\infty(0,+\infty;E)$ is called the *minimal trajectory attractor* (for the trajectory space $\mathcal{H}^+$) if
i) $\mathcal{U}$ is compact in $C([0,+\infty);E_0)$ and bounded in $L_\infty(0,+\infty;E)$;
ii) $T(t)\mathcal{U} = \mathcal{U}$ for any $t \geq 0$;
iii) $\mathcal{U}$ is attracting;
iv) $\mathcal{U}$ is contained in any other set satisfying conditions i), ii), iii).

A set $\mathcal{A} \subset E$ is called the *global attractor* (in E_0) for the trajectory space $\mathcal{H}^+$ if
i) $\mathcal{A}$ is compact in E_0 and bounded in E;
ii) for any bounded in $L_\infty(0,+\infty;E)$ set $B \subset \mathcal{H}^+$ the attraction property is fulfilled:

$$\sup_{u \in B} \inf_{v \in \mathcal{A}} \|u(t) - v\|_{E_0} \underset{t\to\infty}{\to} 0;$$

iii) $\mathcal{A}$ is the minimal set satisfying conditions i) and ii).

Attractors for our model: basic assumptions

a) Ω is bounded.

b) $m > 2$ (although $m > (d+1)/3$ is enough for the dissipative estimates).

c) $\phi \in L_1, \ \nabla\phi \in L_\infty$.

d) k, χ and f are continuously differentiable functions, $\chi' \geq 0$, $k \geq 0$, $k(0) = 0$.

e) The initial concentration of oxygen does not exceed some constant $c_{\mathcal{O}}$: this unusual assumption is to overcome the presence of steady-state solutions $(n \equiv 0, c \equiv c_0, u \equiv 0)$ with arbitrarily large constants c_0 which impede existence of attractors.

f) There exists a positive number γ so that

$$f(y) + 2\gamma y \leq C, \ \ y \geq 0, \tag{9}$$

g)

$$|f(y)| \leq C(y^m + 1), \ \ y \geq 0. \tag{10}$$

Existence of attractors

Let

$$E = L_{m/2} \times H^1 \times H$$

and

$$E_0 = W_{m/2}^{-\delta} \times H^{1-\delta} \times V_\delta^*,$$

where $\delta \in (0,1]$ is a fixed number. Then the trajectory space $\mathcal{H}^+$ consisting of all admissible weak solutions to (1)–(5) possesses a minimal trajectory attractor and a global attractor in the above sense.

Open problems seeming to be manageable

- *Global existence in the subcritical case for classes of kinetic functions containing* $f = 0$
- *Similar setting in non-Newtonian/viscoelastic fluids, e.g., blood; non-Newtonian effects due to large densities of cells*
- *Attractors for* $m \leq 2$*: technical obstacle is the non-reflexivity of* L_1
- *Does the attractor merely consist of the steady-state solutions, or it is more complex?*
- *Other boundary conditions*

THANK YOU