Models in population dynamics: from discrete to continuous and back

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Workshop on PDE’s and Biomedical Applications — 04 to 06/12/2014

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The SIS model

Classical SIS-ODE:

\[ S + I \xrightarrow{\alpha} 2I; \]
\[ I \xrightarrow{\beta} S. \]

\[ \dot{S} = -\alpha S I + \beta I \]
\[ \dot{I} = \alpha S I - \beta I. \]
Without loss of generality, that $S(0) + I(0) = 1$:

$$I' = \bar{\alpha} I \left(1 - \frac{\bar{\beta}}{\bar{\alpha}} - I\right).$$

(2)

Let $\bar{R}_0 := \bar{\alpha}/\bar{\beta}$.

$$\lim_{t \to \infty} I(t) = \begin{cases} 0, & \bar{R}_0 \leq 1; \\ \ell_* = 1 - \frac{1}{\bar{R}_0}, & \bar{R}_0 > 1. \end{cases}$$
SIS-DTMC

- If he or she is of type I, then it becomes S with probability $\beta$;
- If he or she is of type S, then it becomes I with probability proportional to the number of infected in the population: $\alpha x$.

\[
T^+(x) = \alpha x(1 - x), \\
T^0(x) = 1 - T^+(x) - T^-(x), \\
T^-(x) = x\beta.
\]
Let $P_{(N,\Delta t)}(x, t)$ be the probability to find a fraction $x$ of $I$ individuals at time $t$ in a population of size $N$, evolving in time steps of size $\Delta t$.

The master equation is:

$$P_{(N,\Delta t)}(x, t + \Delta t) = T^+(x - z)P_{(N,\Delta t)}(x - z, t)$$
$$+ T^0(x)P_{(N,\Delta t)}(x, t)$$
$$+ T^-(x + z)P_{(N,\Delta t)}(x + z, t).$$
Let \( p_{(N,\Delta t)}(x, t) = NP_{(N,\Delta t)}(x, t) \), and \( \epsilon = 1/N \).

If \( p_{(N,\Delta t)} \) is a smooth function then it need to satisfy

\[
\partial_t p = -\partial_x \left\{ x \left[ R_0(1-x) - 1 \right] p \right\} + \frac{\epsilon}{2} \partial_x^2 \left\{ x(R_0(1-x) + 1)p \right\} + O(\epsilon^2),
\]

\[
\frac{\epsilon}{2} \left( (1 - R_0)p(1, t) + \partial_x p(1, t) \right) + p(1, t) = 0
\]

**Theorem**

For smooth initial conditions, Lax-Richtmyer equivalence theorem implies that \( p_{(N,\Delta t)} \to p \) pointwise, as \( \epsilon \to 0 \), where \( p \) satisfies

\[
\partial_t p + \partial_x \left\{ x \left[ R_0(1-x) - 1 \right] p \right\} = 0 \quad PDE \text{ form of SIS-ODE}
\]

\[
p(1, t) = 0.
\]
Write
\[ u(x, t) = x [R_0(1 - x) - 1] p(x, t) \]
Then \( u \) satisfies
\[ \partial_t u = x [R_0(1 - x) - 1] \partial_x u. \]
Hence
\[ p(x, t) = \frac{\phi_{-t}(x) [R_0(1 - \phi_{-t}(x)) - 1]}{x [R_0(1 - x) - 1]} p_0(\phi_{-t}(x)) \]
where \( \phi_t \) is the flowmap for the SIS-ODE.
Want to keep finite population effects — do not send $\epsilon \to 0$

Look for solution to

$$
\partial_t p = -\partial_x \{x [R_0(1 - x) - 1] p\} + \frac{\epsilon}{2} \partial_x^2 \{x (R_0(1 - x) + 1) p\} + O(\epsilon^2),
$$

$$
\frac{\epsilon}{2} ((1 - R_0) p(1, t) + \partial_x p(1, t)) + p(1, t) = 0
$$

$p(x, 0) = p^0(x);$  

$$
\frac{d}{dt} \int_0^1 p(x, t) \, dx = 0;
$$

$$
\lim_{t \to \infty} p(\cdot, t) = \delta_0.
$$
\[
\int_0^\infty \int_0^1 p(x, t) \partial_t g(x, t) \, dx \, dt + \\
\frac{\epsilon}{2} \int_0^\infty \int_0^1 p(x, t) x (R_0(1 - x) + 1) \, dx \, dt \\
\int_0^\infty \int_0^1 p(x, t) x (R_0(1 - x) - 1) \, dx \, dt + \\
\int_0^1 p(x, 0) g(x, 0) \, dx = 0,
\]

where
\[
g \in C^\infty_c([0, 1] \times [0, \infty)), \quad \partial_x g(1, t) = 0.
\]
Proposition

If \( p \in L^\infty([0, \infty); BM^+([0, 1])) \) is a solution to (3) then

\[
\frac{d}{dt} \int_0^1 p(x, t) \, dx = 0.
\]
Theorem

Let $p^0(x) \in \mathcal{BM}^+(\mathbb{R})$. Then Equation (3) has a unique solution $p \in L^\infty([0, \infty); \mathcal{BM}^+([0, 1]))$, that is given by

$$p(x, t) = a(t)\delta_0 + r(x, t)$$

where $r$ satisfies

$$\partial_t r = -\partial_x \left\{ x \left[ R_0(1 - x) - 1 \right] r + \frac{\epsilon}{2} \partial_x^2 \left\{ x(R_0(1 - x) + 1)r \right\} \right\},$$

$$\frac{\epsilon}{2} \left( (1 - R_0)r(1, t) + \partial_x r(1, t) \right) + r(1, t) = 0$$

$$r(x, 0) = r_0 + b_0\delta_1$$

and

$$a(t) = \frac{\epsilon(R_0 + 1)}{2} \int_0^t r(0, s) \, ds + a_0.$$
Proposition

\[ \lim_{t \to \infty} r(x, t) = 0 \quad \text{and} \quad \lim_{t \to \infty} a(t) = 1. \]

In particular,

\[ \lim_{t \to \infty} p(\cdot, t) = \delta_0 \]
Probability that the disease dies out — $(S_0, I_0) = (199, 1)$

- $R_0 = 1$ (blue line)
- $R_0 = 2$ (green line)
- $R_0 = 4$ (red line)

Graph shows the probability over time $t$. Initial conditions $(S_0, I_0)$.
For the Moran process an inverse numerical analysis argument yields

\[- \int_0^\infty \int_0^1 p(x, t) \partial_t g(x, t) \, dx \, dt - \int_0^1 p(x, t_0) g(x, t_0) \, dx = \kappa \int_0^\infty \int_0^1 p(x, t) \left( x(1-x) \partial^2 g(x, t) \right) \, dx \, dt \]

\[+ \int_0^\infty \int_0^1 p(x, t) \left[ x \left( \psi(x) - \bar{\psi}(x) \right) \partial g(x, t) \right] \, dx \, dt. \tag{4}\]

\[p(x, t) = a(t) \delta_0 + q(x, t) + b(t) \delta_1, \text{ with } q \text{ being the unique } L^2 \text{ classical solution to (4)}. \]
Also

\[
\lim_{t \to \infty} q(x, t) = 0.
\]

Additionally, we have the following conservation laws:

\[
\frac{d}{dt} \int_0^1 p(x, t) \, dx = 0 \quad \text{and} \quad \frac{d}{dt} \int_0^1 \phi(x)p(x, t) \, dx = 0,
\]

\[
\phi'' + x \left(\psi(x) - \bar{\psi}(x)\right) \phi' = 0, \quad \phi(0) = 0, \quad \phi(1) = 1.
\]
Results extended to multidimensional Wright-Fisher process via measure solutions.

Similar derivation for infinite population limit of SIR model.

Derivation of three Holling functional responses for Lotka type predator-prey models.

Approximation of slow manifold via center-manifold theory in other scalings.

Multiscale reduction for Wolbachia models recovering invasion thresholds by Turelli et al.
References

F.A.C.C. Chalub & MOS
Discrete versus continuous models in evolutionary dynamics: from simple to simpler — and even simpler — models.

F.A.C.C. Chalub & MOS
A non-standard evolution problem arising in population genetics.

F.A.C.C. Chalub & MOS
From discrete to continuous evolution models: a unifying approach to drift-diffusion and replicator dynamics.

F.A.C.C. Chalub & MOS
The SIR epidemic model from a PDE point of view.
J.H.P. Dawes & MOS
A derivation of Holling's type I, II and III functional responses in predator-prey systems.


F.A.C.C. Chalub & MOS
The frequency-dependent Wright-Fisher model: diffusive and non-diffusive approximations.


F.A.C.C. Chalub & MOS
Discrete and continuous SIS epidemic models: a unifying approach.

Thank You!