ON THE REGULARITY, THE STABILITY AND OPTIMAL CONTROL OF THE FREE BOUNDARY IN 2-PHASES HETEROGENEOUS STATIONARY PB.S

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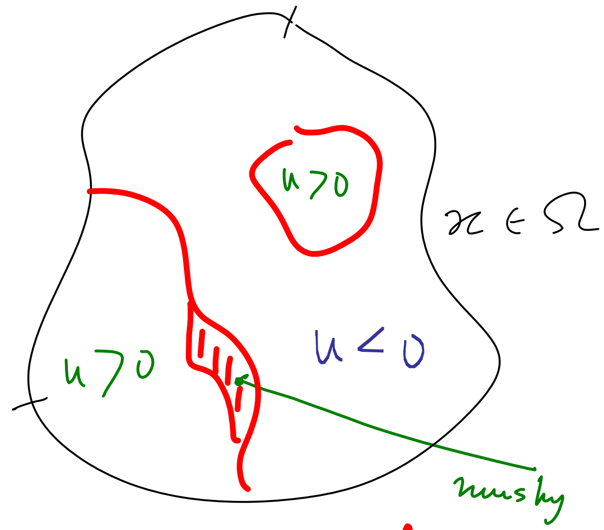
$$A u + \zeta = f \quad \text{in } \Omega \subset \mathbb{R}^n, \text{ bd.}$$

(*)

$$A u = -\operatorname{div}(a(x, Du)), \quad f = f(x)$$

$$\underline{\lambda}_\pm = \underline{\lambda}_\pm(x) \geq 0$$



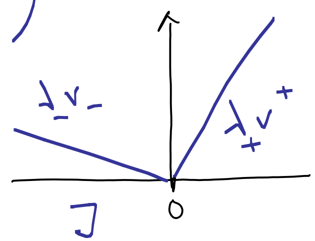
(ζ) a.e. in Ω
$$\begin{cases} \zeta = \lambda_+ \chi_+ - \lambda_- \chi_- \in \partial J(u) \\ 0 \leq \chi_{\{u>0\}} \leq \chi_+ \leq 1 - \chi_{\{u<0\}} \end{cases}$$

$\Phi_0 = \{u = 0\}$ free boundary

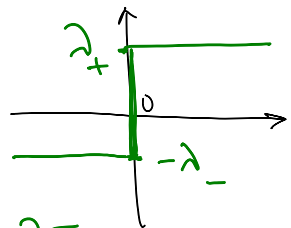
$\Phi = \partial\{u>0\} \cup \partial\{u<0\}$

$$J(v) = \int_\Omega \lambda_+ v^+ + \int_\Omega \lambda_- v^- \qquad (v = v^+ - v^-)$$

$$\partial J(u) = \{\zeta : J(v) - J(u) \geq \langle \zeta, v-u \rangle, \ \forall v\}$$



J



∂J

Quasilinear 2nd order elliptic operator of Δ_p-type

$\kappa \in [0,1]$

$$\begin{cases} Au \simeq -\operatorname{div}\big(\underbrace{M(x)\left(\kappa+|Du|^2\right)^{\frac{p-2}{2}}Du}_{a(x,Du)}\big) \\ \dfrac{\partial u}{\partial \nu_A} = a(Du)\cdot \vec{n}\Big|_{\Gamma_N} = g, \quad u\Big|_{\Gamma_D} = h \end{cases}$$

- $\cdot\ a(x,0) = 0$
- $\cdot\cdot\ \sum_{i,j} \frac{\partial a_i}{\partial \eta_j}\xi_i\xi_j \geq c\left(\kappa+|\eta|^2\right)^{\frac{p-2}{2}}|\xi|^2$
- $\therefore\ \sum_{i,j}\left|\frac{\partial a_i}{\partial \eta_j}\right| \leq c\left(\kappa+|\eta|^2\right)^{\frac{p-2}{2}}$
- $\therefore\ |a(x,\eta)-a(y,\eta)| \leq C|x-y||\eta|^{p-1}$

$$\langle Au, v\rangle = \int_\Omega a(Du)\cdot Dv \ .$$

$$\langle L, v\rangle = \int_\Omega fv + \int_{\Gamma_N} gv \ ; \quad V_h = \{v \in W^{1,p}(\Omega) : v = h \text{ on } \Gamma_D\}, \quad 1<p<\infty$$

Thm ($\exists!$). $\cdot$ For $h \in W^{1,p}(\Omega)$, $f \in L^\infty(\Omega)$, $g \in L^p(\Gamma_N)$ there exists a unique solution to the variational inequality problem

$$u \in V_h : \ \langle Au, v-u\rangle + J(v) - J(u) \geq \langle L, v-u\rangle, \quad \forall v \in V_h$$

$\cdot\cdot$ If $h_k \xrightarrow{W^{1,p}} h$, $g_k \xrightarrow{L^p} g$, $f_k \xrightarrow{L^p} f$ then $u_k \to u$ in $W^{1,p}(\Omega)$.

NOTE. $Au = f - \lambda_+\chi_+ + \lambda_-\chi_- \in L^\infty(\Omega)$; $u \in C^{1,\alpha}_{loc}$ and solves **(*) (S)**

Other choices of the operator A

$$Au = -\Delta_{p(x)} u = -\nabla\cdot\left(|\nabla u|^{p(x)-2}\nabla u\right)$$

$p(x)$ - Laplacian

$1 < \underline{p} \le p(x) \le \overline{p} < \infty$

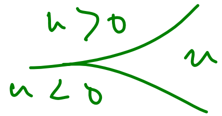
p Lipschitz continuous

$$A = -\Delta \quad \text{(Laplacian)}$$

- If $A = -\Delta$ and $f = 0$ it has shown that $u \in C^{1,1}(\Omega)$ for $\lambda_\pm$ constants (Uraltseva, 2001) and $\lambda_\pm \in C^{0,1}$ (Shahgholian, etc.)

- • $A = -\Delta$, $f = 0$, $\lambda_\pm \in \mathbb{R}$ then $\mathcal{H}^{n-1}(\partial\{u>0\} \cup \partial\{u<0\}) < \infty$ (Weiss - 2001) and (!).

When $f = 0$ we may have a situation



(Uraltseva-Shahgholian-Weiss)

- • • $-\Delta_p$, $\lambda_\pm \in \mathbb{R}$ Edquist & Lindgren 2009

2012 Book by Petrosyan, Shahgholian, Uraltseva

Nondegeneracy of the free boundary (null mushy region)

(*) $\quad Au + \lambda_+ \chi_+ - \lambda_- \chi_- = f$ a.e. in Ω $\quad \left(0 \le \chi_{\{u>0\}} \le \chi_+ \le 1 - \chi_{\{u<0\}}\right)$

(#) $\quad \text{meas}\left(\{u=0\}\right) = 0$ $\quad$ (Lebesgue measure in Ω or Q)

Note that $\{0<\chi_\pm<1\} \subset \{u=0\}$. So (#) implies $\text{meas}\left(\{0<\chi_+<1\} \cup \{0<\chi_-<1\}\right) = 0$, which is sufficient to

$$\chi_+ = \chi_{\{u>0\}} \quad \text{and} \quad \chi_- = \chi_{\{u<0\}} \qquad (!)$$

Thm 2 If $A0=0$ and $f > \lambda_+$ or $f < -\lambda_-$ a.e. in Ω then (#) holds!

When we have $Au = 0$ a.e. in $\{u=0\}$, then from (*) we have

$$-\lambda_- \le \lambda_+ \chi_+ - \chi_- \lambda_- = f \le \lambda_+ \qquad \text{a.e. impossible!}$$

NOTE When $f \equiv 0$ it is well known that (#) may in general be violated.

Continuous Dependence of the Phases +/−

Thm 3 ($\lambda_\pm > 0$) Let $f_k \underset{k}{\to} f$, $g_k \underset{k}{\to} g$ and $h_k \underset{k}{\to} h$ s.t. $u_k \to u$ in $W^{1,q}$. Then

$$\chi_{\{u_k \gtrless 0\}} \xrightarrow[k]{} \chi_{\{u \gtrless 0\}} \quad \text{in } L^q(\Omega),\ 1 \le q < \infty,$$

provided **(#)** (nondegeneracy of Φ_0) holds, i.e. we have for u_k and u

$$(*) \qquad Au + \lambda_+ \chi_{\{u>0\}} - \lambda_- \chi_{\{u<0\}} = f \quad \text{a.e. in } \Omega.$$

Proof: Let $\chi_{\{u^k \gtrless 0\}} \underset{k}{\rightharpoonup} \chi^*_\pm$ in $L^\infty(\Omega)$-weakly*. We have

$$0 = \int_\Omega u_k^\mp \chi_{\{u_k \gtrless 0\}} \to \int_\Omega u^\mp \chi^*_\pm = 0, \quad \text{and } \chi^*_\pm = 0 \text{ if } u \lessgtr 0 \quad \text{and}$$

we find $0 \le \chi^*_\pm \le 1 - \chi_{\{u \lessgtr 0\}}$. Since $\int_O u_k^\pm \chi_{\{u_k \gtrless 0\}} \to \int_O u^\pm \chi^*_\pm = \int_O u^\pm$

for arbitrary $O \subset \Omega$, we obtain $\chi_{\{u \gtrless 0\}} \le \chi^*_\pm$. But (#) implies $\chi_{\{u \gtrless 0\}} = 1 - \chi_{\{u \lessgtr 0\}}$

we conclude $\chi^*_\pm = \chi_{\{u \gtrless 0\}}$ and the convergence follows (strongly)

L^1- estimates on the two phases

For our class of operators A, we may extend a lemma by Brezis-Strauss

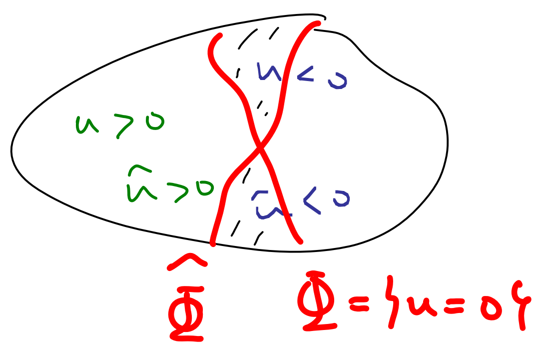
(BS) $\|\zeta - \hat{\zeta}\|_{L^1(\Omega)} \leq \|f - \hat{f}\|_{L^1(\Omega)} + \|g - \hat{g}\|_{L^1(\Gamma_N)}$ $\quad \zeta \in \partial J(u)$, $\hat{\zeta} \in \partial J(\hat{u})$ $\quad h = \hat{h}$

Thm 4 If nondegeneracy **(#)** holds, in particular, $\hat{\chi}_{\pm} = \chi_{\{\hat{u} \gtrless 0\}} = 1 - \chi_{\{u \lessgtr 0\}}$
and $\lambda_+ + \lambda_- \geq \mu > 0$ in Ω, then

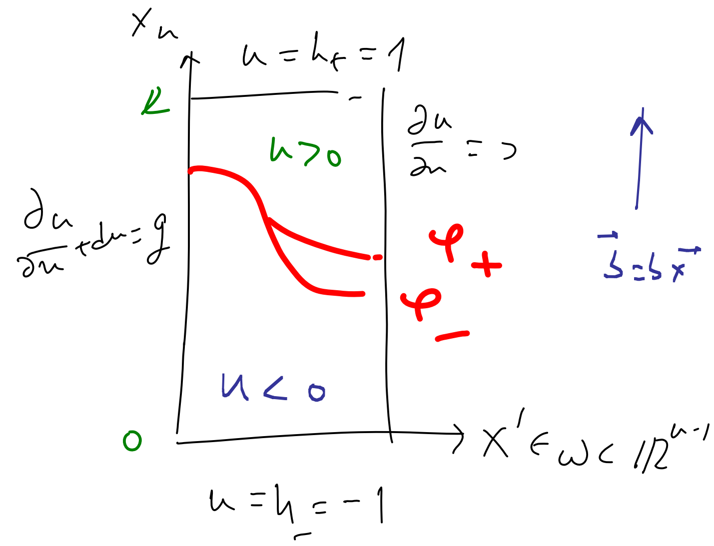
$$\mu \, \mathrm{meas}(\{u>0\} \dot{-} \{\hat{u}>0\}) = \mu \|\chi_{\pm} - \hat{\chi}_{\pm}\|_{L^1(\Omega)} \leq \|f - \hat{f}\|_{L^1(\Omega)} + \|g - \hat{g}\|_{L^1(\Gamma_N)} \qquad (h = \hat{h})$$

to Observation: $\zeta = \lambda_+ \chi_+ - \lambda_- \chi_-$ and $\hat{\zeta} = \lambda_+ \hat{\chi}_+ - \lambda_- \hat{\chi}_-$. Apply

$$|\zeta - \hat{\zeta}| = |\lambda_+ (\chi_+ - \hat{\chi}_+) - \lambda_- (\chi_- - \hat{\chi}_-)|$$
$$= |(\lambda_+ + \lambda_-)(\chi_+ - \hat{\chi}_+)|$$
$$\geq \mu |\chi_+ - \hat{\chi}_+|$$



SPECIAL GEOMETRIES



i) Monotone in x_n : $\nabla_n u \geq 0$

$Au = -\Delta u + b\,\nabla_n u + cu$

$\nabla_n f \geq 0,\ \nabla_n g \geq 0 \qquad \Omega = \omega \times (0, \ell)$

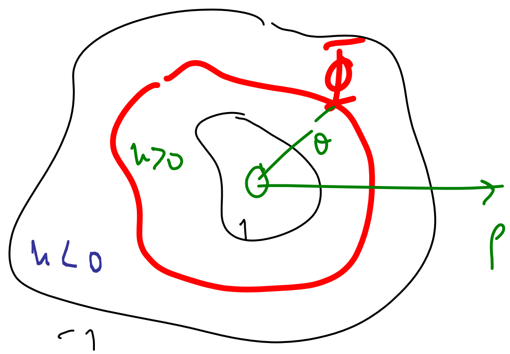
$\varphi_{\pm}(x') = \inf_{\sup} \{ x_n : u(x', x_n) \gtrless 0 \}$

if $\varphi_+ = \varphi_- = \varphi \qquad \int_\Omega |\chi_+ - \hat{\chi}_+| = \int_\omega |\varphi - \hat{\varphi}| \leq \frac{1}{2\lambda} \{ \int_\Omega |f - \hat{f}| + \int_\Gamma |g - \hat{g}| \}$

if also $\|\nabla\varphi\|_{L^\infty} \leq C$ we can estimate the Hölder norm $\|\varphi - \hat{\varphi}\|_{C^{0,\alpha}}$

ii) Starshaped in $\mathbb{R}^n$

with respect to a small ball $B_\delta(0)$:



$\rho = \varphi(\theta) \quad , \theta \in \Theta_{n-1}$

$\partial_\rho \psi \leq 0$ yields $\varphi \in C^{0,1}_\Theta$

REGULARITY OF THE FREE BOUNDARY

$$f, \lambda_\pm \in L^\infty(\Omega) \cap BV_{loc}(\Omega)$$

$$\sum_{i,j} \left| \frac{\partial^2 a_k}{\partial x_i \partial x_j}(x,\eta) \right| \le c\,(k+|\eta|^2)^{\frac{p-1}{2}}$$

$$\sum_{i,j} \left| \frac{\partial^2 a_k}{\partial \eta_i \partial x_j}(x,\eta) \right| \le c\,(k+|\eta|^2)^{\frac{p-2}{2}}$$

Thm 4 Suppose that $u \in C^{1,\alpha}(\Omega) \cap W^{1,p}_{loc}(\Omega)$ is a solution of $Au + \lambda_+ \chi_+ - \lambda_- \chi_- = f$ a.e. in Ω

Then $Au \in BV_{loc}(\Omega)$.

NOTE: The proof uses a remark of Brezis-Kinderlehrer for the obstacle problem. But here $0 \le \chi_{\{u>0\}} \le \chi_+ \le 1-\chi_{\{u<0\}}$

Thm 5 If $(\lambda_+ + \lambda_-) \geq \mu > 0$, and the problem is nondegenerate with $\lambda_+ + \lambda_- \in C^0(\Omega) \cap W^{1,1}_{loc}(\Omega)$,

(#) $\quad \mathbf{meas}(\{u=0\}) = 0 \qquad$ (Lebesgue measure in Ω or Q)

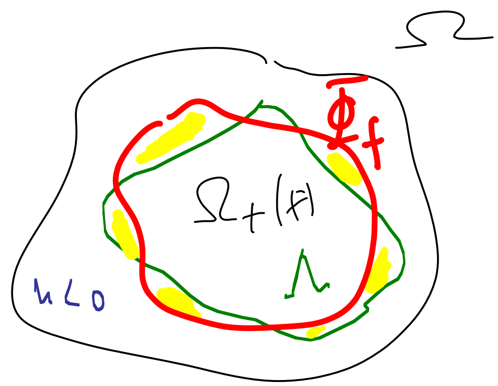
Both $\chi_\pm = \chi_{\{u \gtrless 0\}} \in BV_{loc}$ and the free boundary is, up to a set of null perimeter, the union of at most a countable family of C^1 hypersurfaces.

NOTE: Since $\chi_- = 1 - \chi_+$ a.e., we have

$$\chi_+ = \frac{f - Au + \lambda_-}{\lambda_+ + \lambda_-}$$

OPTIMAL CONTROL OF PHASES

$$I(f) \equiv \int_\Omega |\chi_+(f) - \chi_\Lambda|$$



$\Lambda \subset \Omega$

$$F_1^M \equiv \{ f \in L^1(\Omega) : \|f\|_{L^\infty(\Omega)} \le M \,,\ f \le -\lambda_- \,, f \ge \lambda_+ \text{ in } \Omega \}$$

$\Omega' \subset\subset \Omega$

$M > \|\lambda_\pm\|_{L^\infty}$

$\lambda_+ + \lambda_- \ge \mu > 0$

$$I_\#(f) \equiv I(f) + \int_{\Omega'} |\nabla \chi_+(f)|$$

$$G^{M'} \equiv \{ f \in F_1^M \cap BV_{loc}(\Omega) : \int_{\Omega'} |\nabla f| \le M' \}$$

$$\chi_\Lambda = \begin{cases} 1 & x \in \Lambda \\ 0 & x \notin \Lambda \end{cases}$$

$\chi_+(f) = \chi_{\Omega_+(f)}$

Thm 6 A) Let $\mathcal{U} \subset F_1^M$ be compact for L^1-topology

$$\exists f_* \in \mathcal{U} : I(f_*) \le I(f) \,, \forall f \in \mathcal{U}.$$

$\Omega_+(f)$ Caccioppoli set

B) Let $\lambda_\pm \in C^0(\bar{\Omega}') \cap W^{1,1}(\Omega')$

$$\exists f_\# \in G^{M'} : I_\#(f_\#) \le I_\#(f) \,, \forall f \in G^{M'}$$